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Condensations of $C_p(X)$ onto σ-compact spaces

V. V. Tkachuk *

Abstract. We show, in particular, that if $nw(N_t) \leq \kappa$ for any $t \in T$ and C is a dense subspace of the product $\prod\{N_t : t \in T\}$ then, for any continuous (not necessarily surjective) map $\varphi : C \to K$ of C into a compact space K with $t(K) \leq \kappa$, we have $\Psi(\varphi(C)) \leq \kappa$. This result has several applications in C_p-theory. We prove, among other things, that if K is a non-metrizable Corson compact space then $C_p(K)$ cannot be condensed onto a σ-compact space. This answers two questions published by Arhangel'skii and Pavlov.

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1. Introduction.

A weaker topology on a space X can be considered an approximation of the topology of X. If this approximation has some nice properties then we can obtain a lot of useful information about the space X. Thus it is natural to find out when a space has a weaker compact topology. This is an old topic and an extensive research has been done here both in general topology and descriptive set theory.

The quest for nice condensations of function spaces had its origin in functional analysis after Banach asked whether every separable Banach space has a weaker compact metrizable topology. This problem was solved positively by Pytkeev [9]. Answering a question of Arhangel'skii, Casarrubias–Segura showed in [5] that function spaces of Cantor cubes have a weaker Lindelöf topology but it is consistent that some of them do not have a weaker compact topology.

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Arhangel'skii and Pavlov [4] studied systematically when $C_p(X)$ has a weaker compact topology and formulated some open questions on weaker σ-compact topologies on $C_p(X)$. It is also worth mentioning that Marciszewski [7] gave a consistent example of a space $X \subset \mathbb{R}$ such that $C_p(X)$ does not have a weaker σ-compact topology.

In this paper we consider product spaces $N = \prod_{t \in T} N_t$ such that $nw(N_t) \leq \kappa$ for all $t \in T$. We prove that if C is a dense subspace of N and $\varphi : C \to K$ is a continuous (not necessarily surjective) map of C into a compact space K with $t(K) \leq \kappa$ then $\Psi(\varphi(C)) \leq \kappa$, i.e., every closed subset of $\varphi(C)$ is the intersection of at most κ-many open subsets of $\varphi(C)$. This result has several important applications in C_p-theory. We establish, in particular, that if X is an ω-monolithic space such that $l(C_p(X)) = t(C_p(X)) = \omega$ and $C_p(X)$ condenses onto a σ-compact space then X is cosmic. As a consequence, if X is a non-metrizable Corson compact space then $C_p(X)$ does not condense onto a σ-compact space. This answers Questions 29 and 30 of the paper of Arhangel'skii and Pavlov [4].

Any compact space of countable tightness has countable π-character (see [1, Theorem 2.2.20]). This easily implies that if $C_p(X)$ embeds in such a compact space then X is countable. Therefore it is natural to conjecture that every continuous image of $C_p(X)$ has a countable network whenever it embeds in a compact space of countable tightness. Another reason to believe that this conjecture might be true is a theorem of Tkachenko [12] which states that if a compact space K of countable tightness is a continuous image of a Lindelöf Σ-group then K is metrizable. At the present moment nothing contradicts the hypothesis that if G is a Lindelöf Σ-group and $\varphi : G \to K$ is a continuous map, where K is compact and $t(K) \leq \omega$ then $\varphi(G)$ has a countable network. We prove this conjecture for the spaces $C_p(X)$ with the Lindelöf Σ-property.

2. Notation and terminology.

All spaces under consideration are assumed to be Tychonoff. If X is a space then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X) \setminus \{\varnothing\}$; given an arbitrary set $A \subset X$ let $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. If $x \in X$ then we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. The space $\mathbb{R}$ is the real line with its natural topology and $\mathbb{N} = \omega \setminus \{0\}$. If X and Y are spaces then $C_p(X, Y)$ is the space of real-valued continuous functions from X to Y endowed with the topology of pointwise convergence. We write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. The expression $X \simeq Y$ says that the spaces X and Y are homeomorphic.

A family $\mathcal{N}$ of subsets of a space Z is called *a network* if for any $U \in \tau(Z)$ there is $\mathcal{N}' \subset \mathcal{N}$ such that $\bigcup \mathcal{N}' = U$. The *network weight* $nw(Z)$ of a space Z is the minimal cardinality of a network in Z. A space X is called *cosmic* if the network weight of X is countable. If $x \in X$ then a family $\mathcal{A}$ is a network of X at the point x if $x \in \bigcap \mathcal{A}$ and for any $U \in \tau(x, X)$ there is $A \in \mathcal{A}$ such that $A \subset U$.

A family $\mathcal{B} \subset \tau^*(X)$ is a π-base of X at a point $x \in X$ if for any $U \in \tau(x, X)$ there is $B \in \mathcal{B}$ with $B \subset U$. The minimal cardinality of a π-base of X at x is denoted by $\pi\chi(x,X)$ and $\pi\chi(X) = \sup\{\pi\chi(x,X) : x \in X\}$. If φ is a cardinal invariant then $h\varphi(X) = \sup\{\varphi(Y) : Y \subset X\}$ is the hereditary version of φ. If X is a space and F is a closed subset of X then *pseudocharacter* $\psi(F,X)$ of the set F in the space X is the minimal cardinality of a family $\mathcal{U} \subset \tau(F,X)$ such that $\bigcap \mathcal{U} = F$; let $\psi(X) = \sup\{\psi(\{x\},X) : x \in X\}$ and $\Psi(X) = \sup\{\psi(F,X) : F$ is a closed subset of $X\}$.

The *tightness* $t(X)$ of a space X is the minimal cardinal κ such that, for any $A \subset X$, if $x \in \overline{A}$ then there is $B \subset A$ with $|B| \le \kappa$ such that $x \in \overline{B}$. We use the Russian term *condensation* to denote a continuous bijection. A space Z is called *κ-monolithic* if for any $A \subset Z$ with $|A| \le \kappa$, we have $nw(\overline{A}) \le \kappa$.

If we have a product $Z = \prod_{t\in T} Z_t$ and $A \subset T$ then $Z_A = \prod_{t\in A} Z_t$ is the A-face of Z and $\pi_A : Z \to Z_A$ is the natural projection. A set $F \subset Z$ *depends on* $A \subset T$ if $\pi_A^{-1}\pi_A(F) = F$; if F depends on a set of cardinality $\le \kappa$ then we say that F depends on at most κ-many coordinates. A set $E \subset Z$ *covers a face* Z_A if $\pi_A(E) = Z_A$. Suppose that, for every $t \in T$ we have a family $\mathcal{N}_t$ of subsets of Z_t and let $\mathcal{N} = \{\mathcal{N}_t : t \in T\}$. If we have a faithfully indexed set $A = \{t_1, \dots, t_n\} \subset T$ and $N_i \in \mathcal{N}_{t_i}$ for each $t \le n$ then let $[t_1, \dots, t_n, N_1, \dots, N_n] = \{x \in Z : x(t_i) \in N_i$ for all $i = 1, \dots, n\}$. A set $H \subset Z$ is called *$\mathcal{N}$-standard* (or standard if $\mathcal{N}$ is clear) if $H = [t_1, \dots, t_n, N_1, \dots, N_n]$ for some $t_1, \dots, t_n \in T$ and $N_i \in \mathcal{N}_{t_i}$ for all $i \le n$. In this case we let $\mathrm{supp}(H) = A$ and $r(H) = n$. We also consider that $H = Z$ is the unique standard subset of Z such that $r(H) = 0$. Given any point $x \in Z$ and $A \subset T$ the set $\langle x, A\rangle = \{y \in Z : y(t) = x(t)$ for any $t \in A\}$ is closed in Z. If $A \subset T$ then the face Z_A is called *κ-residual* if $|T \setminus A| \le \kappa$. Say that a non-empty closed set $F \subset K$ is *κ-large* if, for any $x \in F$ and any finite $A \subset T$, the set $\langle x, A\rangle \cap F$ covers a κ-residual face of K.

All other notions are standard and can be found in [6] and [3].

3. Nice continuous images of function spaces.

Our results will be obtained by strengthening a result of Shirokov [11]. Although our modifications of Shirokov's method are minimal, we give complete proofs because the paper [11] has never been translated and, even in Russian, it is completely out of access for a Western reader. In particular, we present the proof of the following lemma established in [11].

Lemma 3.1. *Given an infinite cardinal κ suppose that $nw(N_t) \le \kappa$ for any $t \in T$ and $N = \prod_{t\in T} N_t$. Assume that $C \subset N$ is dense in N, and we have a compact extension K_t of the space N_t for any $t \in T$. If a set $F \subset K = \prod_{t\in T} K_t$ is κ-large then there exists a G_κ-set G in the space K such that $F \subset G$ and $F \cap C = G \cap C$. In particular, $F \cap C$ is a G_κ-subset of C.*

Proof. We can assume, without loss of generality, that $K \setminus F \neq \varnothing$. For every $t \in T$ fix a network $\mathcal{N}_t$ in the space N_t such that $|\mathcal{N}_t| \le \kappa$; we will need the family $\mathcal{M}_t = \{\mathrm{cl}_{K_t}(N) : N \in \mathcal{N}_t\}$. If $\mathcal{M} = \{\mathcal{M}_t : t \in T\}$ then the $\mathcal{M}$-standard subsets of K will be called *standard.* It is easy to see that

(1) the family $\mathcal{H}$ of all standard subsets of K is a network in K at every $x \in C$.

Given standard sets P and P' say that $P' \preceq P$ if $P = [t_1, \ldots, t_n, M_1, \ldots, M_n]$ and there exists a natural $k \le n$ such that $P' = [t_{i_1}, \ldots, t_{i_k}, M_{i_1}, \ldots, M_{i_k}]$ for some distinct $i_1, \ldots, i_k \in \{1, \ldots, n\}$; if $k < n$ then we write $P' \prec P$. We also include here the case when $k = 0$ so $P' = K \preceq P$ for any standard set P. Say that a standard set P is *minimal* if $P \cap F = \varnothing$ but $P' \cap F \neq \varnothing$ whenever $P' \prec P$. It follows from (1) that

(2) for any $x \in C \setminus F$ there exists a minimal standard set P such that $x \in P$.

It will be easy to finish our proof if we establish that

(3) the family $\mathcal{S}$ of minimal standard sets has cardinality not exceeding κ.

Assume, toward a contradiction that $|\mathcal{S}| > \kappa$. Then we can choose $\mathcal{S}_0 \subset \mathcal{S}$ such that $|\mathcal{S}_0| = \kappa^+$ and there exists $n \in \omega$ with $r(P) = n$ for all $P \in \mathcal{S}_0$. Observe first that

(4) if $A \subset T$, a set $D \subset K$ covers the face $K_{T \setminus A}$ and a standard set P is disjoint from D then $\operatorname{supp}(P) \cap A \neq \varnothing$.

Indeed, if $\operatorname{supp}(P) = \{t_1, \ldots, t_k\} \subset T \setminus A$ and $P = [t_1, \ldots, t_k, M_1, \ldots, M_k]$ then it follows from $\pi_{T\setminus A}(D) = K_{T\setminus A}$ that there exists a point $x \in D$ such that $x(t_i) \in M_i$ for all $i \le k$. Therefore $x \in D \cap P$ which is a contradiction.

The set F being κ-large, there exists $A_1 \subset T$ with $|A_1| \le \kappa$ such that F covers the face $K_{T\setminus A_1}$. The property (4) shows that $\operatorname{supp}(P) \cap A_1 \neq \varnothing$ for any $P \in \mathcal{S}_0$. There exists a point $t_1 \in A_1$ such that the family $\mathcal{S}_0' = \{P \in \mathcal{S}_0 : t_1 \in P\}$ has cardinality κ^+. Since $|\mathcal{M}_{t_1}| \le \kappa$, we can find a family $\mathcal{S}_1 \subset \mathcal{S}_0'$ and $M_1 \in \mathcal{M}_{t_1}$ such that $|\mathcal{S}_1| = \kappa^+$ and $[t_1, M_1] \preceq P$ for any $P \in \mathcal{S}_1$.

Proceeding by induction assume that $k < n$ and we have a set $A_k \subset T$ with $|A_k| \le \kappa$ and a family $\mathcal{S}_k$ such that $|\mathcal{S}_k| = \kappa^+$ and, for some $t_1, \ldots, t_k \in A_k$ and $M_i \in \mathcal{M}_{t_i}$ $(i = 1, \ldots, k)$, we have $[t_1, \ldots, t_k, M_1, \ldots, M_k] \preceq P$ for every $P \in \mathcal{S}_k$. Therefore $P = [t_1, \ldots, t_k, s_1, \ldots, s_{n-k}, M_1, \ldots, M_k, E_1, \ldots, E_{n-k}]$ for every $P \in \mathcal{S}_k$; let $Q(P)$ be the set in which s_1 and E_1 are omitted from the definition of P, i.e., $Q(P) = [t_1, \ldots, t_k, s_2, \ldots, s_{n-k}, M_1, \ldots, M_k, E_2, \ldots, E_{n-k}]$. It is clear that $Q(P) \prec P$; since P is minimal, the set $Q(P)$ intersects F for each $P \in \mathcal{S}_k$.

Fix a set $R \in \mathcal{S}_k$ and let $F' = F \cap Q(R)$. The set F being κ-large, we can find $A \subset T$ with $|A| \le \kappa$ such that F' covers the face $K_{T\setminus A}$ and hence the face $K_{T\setminus(A\cup A_k)}$ as well. Let $A_{k+1} = A \cup A_k$ and observe that every set $P \in \mathcal{S}_k$ is disjoint from F'; this, together with (4) shows that $\operatorname{supp}(P) \cap A_{k+1} \neq \varnothing$. Suppose for a moment that $P = [t_1, \ldots, t_k, s_1, \ldots, s_{n-k}, M_1, \ldots, M_k, E_1, \ldots, E_{n-k}] \in \mathcal{S}_k$ and $\{s_1, \ldots, s_{n-k}\} \cap A_{k+1} = \varnothing$. Since F' covers the face $K_{T\setminus A_{k+1}}$, we can find a point $x \in F'$ such that $x(s_i) \in E_i$ for all $i \le n-k$; since also $x(t_i) \in M_i$ for all $i \le k$ because $x \in Q(R)$, we conclude that $x \in F' \cap P$. This contradiction implies that $\{s_1, \ldots, s_{n-k}\} \cap A_{k+1} \neq \varnothing$ and hence the set $\operatorname{supp}(P) \setminus \{t_1, \ldots, t_k\}$ intersects the set A_{k+1} for any $P \in \mathcal{S}_k \setminus \{R\}$.

Therefore we can choose a family $\mathcal{S}_{k+1} \subset \mathcal{S}_k$ of cardinality κ^+ together with a point $t_{k+1} \in A_{k+1} \setminus \{t_1, \ldots, t_k\}$ and a set $M_{k+1} \in \mathcal{M}_{t_{k+1}}$ such that we have $[t_1, \ldots, t_{k+1}, M_1, \ldots, M_{k+1}] \preceq P$ for any $P \in \mathcal{S}_{k+1}$. As a consequence, our inductive procedure can be continued to construct a family $\mathcal{S}_n \subset \mathcal{S}$ such that

$|\mathcal{S}_n| = \kappa^+$ while $[t_1, \ldots, t_n, M_1, \ldots, M_n] \preceq P$ for any $P \in \mathcal{S}_n$. Recalling that $r(P) = n$, we conclude that we have the equality $P = [t_1, \ldots, t_n, M_1, \ldots, M_n]$ for each $P \in \mathcal{S}_n$; this contradiction shows that $|\mathcal{S}| \leq \kappa$, i.e., (3) is proved.

It is straightforward that $G = K \setminus (\bigcup \mathcal{S})$ is a G_κ-subset of K such that $F \subset G$ and $F \cap C = G \cap C$. □

The following result generalizes Theorem 1 of [11].

Theorem 3.2. *Given an infinite cardinal κ suppose that $nw(N_t) \leq \kappa$ for any $t \in T$ and $C \subset N = \prod_{t \in T} N_t$ is a dense subspace of N. Assume additionally that we have a continuous (not necessarily surjective) map $\varphi : C \to L$ of C into a compact space L. If $y \in C' = \varphi(C)$ and $h\pi\chi(y, L) \leq \kappa$ then $\psi(y, C') \leq \kappa$.*

Proof. There is no loss of generality to assume that C' is dense in L. Choose a compact extension K_t of the space N_t for any $t \in T$; then $K = \prod_{t \in T} K_t$ is a compact extension of both N and C. There exist continuous maps $\Phi : \beta C \to L$ and $\xi : \beta C \to K$ such that $\Phi|C = \varphi$ and $\xi(x) = x$ for any $x \in C$. It is clear that both Φ and ξ are surjective.

For every $t \in T$ fix a network $\mathcal{N}_t$ in the space N_t such that $|\mathcal{N}_t| \leq \kappa$ and let $\mathcal{M}_t = \{\mathrm{cl}_{K_t}(N) : N \in \mathcal{N}_t\}$. If $\mathcal{M} = \{\mathcal{M}_t : t \in T\}$ then the $\mathcal{M}$-standard subsets of K will be called *standard.* Our first step is to prove that

(5) the set $F_y = \xi(\Phi^{-1}(y))$ is κ-large.

Fix a point $x \in F_y$, a finite $A \subset T$ and consider the set $P = \langle x, A \rangle = \{x' \in K : x'(t) = x(t)$ for all $t \in A\}$. It follows from $P \cap F_y \neq \varnothing$ that $\xi^{-1}(P) \cap \Phi^{-1}(y) \neq \varnothing$ and hence $y \in Q = \Phi(\xi^{-1}(P))$. The set Q is compact and it follows from $h\pi\chi(y, L) \leq \kappa$ that we can choose a π-base $\mathcal{B}$ of the space Q at the point y such that $|\mathcal{B}| \leq \kappa$. For every $B \in \mathcal{B}$ pick a set $O_B \in \tau(L)$ such that $\varnothing \neq O_B \cap Q \subset \overline{O}_B \cap Q \subset B$. It follows in a standard way from $c(K) \leq \kappa$ that

(6) for any $U \in \tau^*(L)$, the set $\mathrm{cl}_K(\varphi^{-1}(U))$ depends on at most κ-many coordinates and coincides with the set $\xi(\mathrm{cl}_{\beta C}(\Phi^{-1}(U)))$.

Apply (6) to find a set $S \subset T$ of cardinality at most κ for which $A \subset S$ and the set $D_B = \xi(\mathrm{cl}_{\beta C}(\Phi^{-1}(O_B)))$ depends on S for any $B \in \mathcal{B}$. The face $K_{T \setminus S}$ is residual; to show that $P \cap F_y$ covers $K_{T \setminus S}$ fix any point $w \in K_{T \setminus S}$ and consider the set $E = \{z \in K : \pi_{T \setminus S}(z) = w$ and $\pi_S(z) \in \pi_S(P)\}$. Clearly, E is a non-empty compact subset of P.

Fix any $B \in \mathcal{B}$; it follows from $O_B \cap Q \neq \varnothing$ that there is a point $u \in \xi^{-1}(P)$ such that $\Phi(u) \in O_B$; thus $u \in \Phi^{-1}(O_B)$ which shows that $\xi(u) \in D_B \cap P$. Define a point $u' \in K$ by the equalities $\pi_{T \setminus S}(u') = w$ and $\pi_S(u') = \pi_S(\xi(u))$. Since the sets D_B and P depend on S, we conclude that $u' \in D_B \cap P$. On the other hand, $\pi_S(u') \in \pi_S(P)$ so $u' \in E$, and therefore $E \cap D_B \neq \varnothing$.

As a consequence, $\Phi(\xi^{-1}(E)) \cap \overline{O}_B \neq \varnothing$ and hence $\Phi(\xi^{-1}(E)) \cap B \neq \varnothing$ for any $B \in \mathcal{B}$; since $\Phi(\xi^{-1}(E))$ is a closed subset of Q and $\mathcal{B}$ is a π-base of Q at y, we must have $y \in \Phi(\xi^{-1}(E))$ which implies that $\xi^{-1}(E) \cap \Phi^{-1}(y) \neq \varnothing$ and hence $E \cap F_y \neq \varnothing$. If $v \in E \cap F_y$ then $w = \pi_{T \setminus S}(v) \in \pi_{T \setminus S}(P \cap F_y)$; the point $w \in K_{T\setminus S}$ was chosen arbitrarily so $P \cap F_y$ covers $K_{T\setminus S}$ and hence (5) is proved.

By Lemma 3.1 there exists a G_κ-set G in the space K such that $F_y \subset G$ and $G \cap C = F_y \cap C = \varphi^{-1}(y)$. Therefore we can choose a family $\mathcal{F}$ of compact subsets of K such that $|\mathcal{F}| \leq \kappa$ and $C \setminus F_y \subset \bigcup \mathcal{F} \subset K \setminus F_y$. For any $F \in \mathcal{F}$ the set $W_F = L \setminus \Phi(\xi^{-1}(F))$ is an open neighbourhood of y in L and it is straightforward that $H = \bigcap\{W_F : F \in \mathcal{F}\}$ is a G_κ-subset of L such that $H \cap C' = \{y\}$. □

Corollary 3.3. *Suppose that C is a dense subspace of a product $N = \prod_{t\in T} N_t$ such that $nw(N_t) \leq \kappa$ for each $t \in T$. Assume that K is a compact space with $t(K) \leq \kappa$ and $\varphi : C \to K$ is a continuous (not necessarily surjective) map; let $C' = \varphi(C)$. Then every closed subspace of C' is a G_κ-set, i.e., $\Psi(C') \leq \kappa$; in particular, $\psi(C') \leq \kappa$.*

Proof. Fix a non-empty closed set F' in the space C' and let $F = \mathrm{cl}_K(F')$. Consider the quotient map $p : K \to K_F$ obtained by contracting the set F to a point and let $q = p|C'$. It is easy to see that we have the inequalities $t(K_F) \leq t(K) \leq \kappa$; denote by y the point of the space K_F represented by F and let $C'' = p(C')$. It follows from [1, Theorem 2.2.20] that $h\pi\chi(y, K_F) \leq \kappa$ so Theorem 3.2, applied to the map $p \circ \varphi$, implies that $\psi(y, C'') \leq \kappa$. Since $F' = q^{-1}(y)$, we conclude that F' is a G_κ-subset of C'. □

Corollary 3.4. *Suppose that C is a dense subspace of a product $N = \prod_{t\in T} N_t$ such that $nw(N_t) \leq \kappa$ for each $t \in T$. Assume additionally that $l(C) \leq \kappa$ and K is a compact space with $t(K) \leq \kappa$ such that there exists a continuous (not necessarily surjective) map $\varphi : C \to K$. If $C' = \varphi(C)$ then $hl(C') \leq \kappa$.*

Proof. We have $l(C') \leq \kappa$ while every closed subspace of the space C' is a G_κ-set by Corollary 3.3. Now, a standard proof shows that $hl(C') \leq \kappa$. □

Corollary 3.5. *If C is a dense subspace of a product of cosmic spaces and K is a compact space then, for any continuous map $\varphi : C \to K$, we have $\Psi(\varphi(C)) \leq t(K)$.*

The last corollary has several applications in C_p-theory. Let us start with the following observation.

Proposition 3.6. *(Folklore). If the space of a topological group G embeds in a compact space of countable tightness then G is metrizable. In particular, if $C_p(X)$ embeds in a compact space of countable tightness then $C_p(X)$ is second countable and hence X is countable.*

Proof. Assume that G is a dense subspace of a compact space K with $t(K) \leq \omega$. Then $\pi\chi(g, G) = \pi\chi(g, K) \leq \omega$ (see [1, Theorem 2.2.20]) and hence we have the equality $\chi(g, G) = \pi\chi(g, G) = \omega$ for any $g \in G$ (see [2, Proposition 1.1]) so G is metrizable. □

The following result is a curious generalization of Proposition 3.6 for the case of condensations.

Corollary 3.7. *For any X, the space $C_p(X)$ condenses onto a space embeddable in a compact space of countable tightness if and only if $C_p(X)$ condenses onto a second countable space.*

Proof. Apply Corollary 3.5 and the equality $\psi(C_p(X)) = iw(C_p(X))$. □

However, it would be interesting to find out whether any continuous image of $C_p(X)$ embeddable in a compact space of countable tightness has to be cosmic or even metrizable. It follows from Corollary 3.3 that such an image is a perfect space. The following theorem shows that this conjecture is true when $C_p(X)$ is a Lindelöf Σ-space.

Theorem 3.8. *Suppose that $\varphi : C_p(X) \to K$ is a continuous (not necessarily surjective) map and K is a compact space with $t(K) \le \omega$; let $Y = \varphi(C_p(X))$. Then*

(i) *Y is a perfect space of countable π-weight;*
(ii) *if $C_p(X)$ is a Lindelöf Σ-space then Y is cosmic.*

Proof. That Y is perfect is an immediate consequence of Corollary 3.3. Since ω_1 is a precaliber of $C_p(X)$, it has to be also a precaliber of Y and hence of $\overline{Y}$. The space $\overline{Y}$ being compact, the cardinal ω_1 is a caliber $\overline{Y}$; it follows from $t(\overline{Y}) \le \omega$ that $\overline{Y}$ has a point-countable π-base [10]. This implies that $\pi w(\overline{Y}) = \omega$ and hence $\pi w(Y) = \omega$ as well, i.e., we settled (i).

If $C_p(X)$ is a Lindelöf Σ-space then $C_p(X) \times C_p(X)$ is Lindelöf. The space $Y \times Y$ is a continuous image of $C_p(X) \times C_p(X) \simeq C_p(X \oplus X)$ so we can apply Corollary 3.4 to convince ourselves that $Y \times Y$ is hereditarily Lindelöf and hence Y condenses onto a second countable space. This, together with the Lindelöf Σ-property of Y implies that $nw(Y) \le \omega$ and hence (ii) is proved. □

In the sequel we will need the following lemma from [13].

Lemma 3.9. *If $C_p(X) = \bigcup_{n\in\omega} F_n$ and every F_n is closed in $C_p(X)$ then there exists $n \in \omega$ such that $C_p(X)$ embeds in F_n.*

Theorem 3.10. *Suppose that $l(X^n) = \omega$ for all $n \in \mathbb{N}$ and $C_p(X)$ is Lindelöf. If $C_p(X)$ condenses onto a σ-compact space Y then the space X is separable and $\psi(Y) = \omega$.*

Proof. Fix a condensation $\varphi : C_p(X) \to Y$ and a family $\{K_n : n \in \omega\}$ of compact subsets of Y such that $Y = \bigcup_{n\in\omega} K_n$. The set $F_n = \varphi^{-1}(K_n)$ is closed in $C_p(X)$ for every $n \in \omega$. If $n \in \omega$ and S is an uncountable free sequence in K_n then $S' = \varphi^{-1}(S)$ is an uncountable free sequence in F_n which is impossible because $l(F_n) \le l(C_p(X)) = \omega$ and $t(F_n) \le t(C_p(X)) = \omega$. This contradiction shows that K_n has no uncountable free sequences and therefore $t(F_n) \le \omega$ for any $n \in \omega$.

Apply Lemma 3.9 to see that there exists $n \in \omega$ such that $C \simeq C_p(X)$ for some $C \subset F_n$. Since $\varphi|C$ maps C into K_n, Corollary 3.5 shows that $\psi(\varphi(C)) \le \omega$. Since $\varphi|C$ is a condensation, we have $\psi(C_p(X)) = \psi(C) \le \psi(\varphi(C)) = \omega$ and hence $d(X) = \psi(C_p(X)) = \omega$, i.e., X is separable as promised.

It follows from $\psi(C_p(X)) = \omega$ that $C_p(X) \setminus \{f\}$ is an F_σ-set for any $f \in C_p(X)$. The space $C_p(X)$ being Lindelöf, $C_p(X) \setminus \{f\}$ is Lindelöf as well. Therefore $Y \setminus \{y\}$ is Lindelöf for any $y \in Y$; this implies that $\psi(Y) \leq \omega$. □

Corollary 3.11. *Suppose that X is an ω-monolithic space such that $C_p(X)$ is Lindelöf and X^n is Lindelöf for any $n \in \mathbb{N}$. If $C_p(X)$ condenses onto a σ-compact space Y then $nw(X) = nw(Y) = \omega$.*

Proof. Theorem 3.10 shows that the space X must be separable so $nw(X) = \omega$ by ω-monolithity of X. Therefore $nw(Y) \leq nw(C_p(X)) = nw(X) = \omega$. □

The following result gives a complete answer (in a much stronger form) to Problems 29 and 30 from the paper [4].

Corollary 3.12. *If X is an ω-monolithic compact space such that $C_p(X)$ is Lindelöf and can be condensed onto a σ-compact space then X is metrizable. In particular, if X is a non-metrizable Corson compact space then $C_p(X)$ does not condense onto a σ-compact space.*

Corollary 3.13. *Under MA+¬CH if K is a compact space such that $C_p(K)$ is Lindelöf and can be condensed onto a σ-compact space then X is metrizable.*

Proof. It is a result of Reznichenko (see [3, Theorem IV.8.7]) that MA+¬CH together with the Lindelöf property $C_p(K)$ implies that K is ω-monolithic so we can apply Corollary 3.12 to see that K is metrizable. □

Corollary 3.14. *Assume that $C_p(X)$ is a Lindelöf Σ-space and there exists a condensation of $C_p(X)$ onto a σ-compact space Y. Then $nw(X) = nw(Y) = \omega$.*

Proof. Denote by υX the Hewitt realcompactification of X. It is evident that $C_p(X)$ is a continuous image of the space $C_p(\upsilon X)$. Besides, $Z = \upsilon X$ is a Lindelöf Σ-space by [8, Corollary 3.6] ; since $C_p(Z)$ is also a Lindelöf Σ-space (see [14, Theorem 2.3]), Corollary 3.11 is applicable to Z and we can conclude that $nw(Z) = nw(Y) = \omega$. Since $X \subset Z$, we have $nw(X) \leq nw(Z) = \omega$. □

4. Open problems.

There are still many opportunities for discovering interesting facts about condensations of function spaces. The list below shows some possible lines of research in this direction.

Problem 4.1. *Suppose that K is a compact space of countable tightness and $\varphi : C_p(X) \to K$ is a continuous map. Is it true that $\varphi(C_p(X))$ is cosmic or even metrizable?*

Problem 4.2. *Suppose that $C_p(X)$ is Lindelöf, K is a compact space of countable tightness and $\varphi : C_p(X) \to K$ is a continuous map. Is it true that $\varphi(C_p(X))$ is cosmic or even metrizable?*

Problem 4.3. *Suppose that $C_p(X)$ is hereditarily Lindelöf, K is a compact space of countable tightness and $\varphi : C_p(X) \to K$ is a continuous map. Is it true that $\varphi(C_p(X))$ is cosmic or even metrizable?*

Problem 4.4. *Suppose that X is compact, K is a compact space of countable tightness and $\varphi : C_p(X) \to K$ is a continuous map. Is it true that $\varphi(C_p(X))$ is cosmic or even metrizable?*

Problem 4.5. *Is it true that, for any cardinal κ and any compact space K with $t(K) \leq \omega$, if $\varphi : \mathbb{R}^\kappa \to K$ is a continuous map then $\varphi(\mathbb{R}^\kappa)$ is cosmic or even metrizable?*

Problem 4.6. *Suppose that K is a compact space of countable tightness, G is a topological group with the Lindelöf Σ-property and $\varphi : G \to K$ is a continuous map. Is it true that $\varphi(G)$ is cosmic?*

Problem 4.7. *Suppose that $C_p(X)$ is Lindelöf and there exists a condensation of $C_p(X)$ onto a σ-compact space Y. Must Y be cosmic?*

Problem 4.8. *Suppose that $C_p(X)$ is Lindelöf and there exists a condensation of $C_p(X)$ onto a σ-compact space Y. Must X be separable?*

Problem 4.9. *Suppose that $C_p(X)$ condenses onto a space of countable π-weight. Must X be separable?*

Problem 4.10. *Suppose that $C_p(X)$ is Lindelöf and $\varphi : C_p(X) \to Y$ is a continuous onto map. Is it true that every compact subspace of Y has countable tightness?*

Problem 4.11. *Suppose that K is Eberlein compact and $\varphi : C_p(K) \to Y$ is a continuous surjective map of $C_p(X)$ onto a σ-compact space Y. Must Y be cosmic?*

Problem 4.12. *Suppose that K is Corson compact and $\varphi : C_p(K) \to Y$ is a continuous surjective map of $C_p(X)$ onto a σ-compact space Y. Must Y be cosmic?*

Problem 4.13. *Suppose that X is a space such that $C_p(X)$ has the Lindelöf Σ-property and $\varphi : C_p(K) \to Y$ is a continuous surjective map of $C_p(X)$ onto a σ-compact space Y. Must Y be cosmic?*

Problem 4.14. *Suppose that $C_p(X)$ is Lindelöf and X^n is also Lindelöf for any $n \in \mathbb{N}$. Assume additionally that there exists a condensation of $C_p(X)$ onto a σ-compact space Y. Must Y be cosmic?*

Problem 4.15. *Suppose that K is a perfectly normal compact space. Is it true that every σ-compact continuous image of $C_p(X)$ has a countable network?*

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Vladimir V. Tkachuk (vova@xanum.uam.mx)
Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco, 186, Col. Vicentina, Iztapalapa, C.P. 09340, México D.F.