Applied General Topology
© Universidad Politécnica de Valencia
Volume 4, No. 2, 2003
pp. 211–216

Unusual and bijectively related manifolds

John G. Hocking

Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. A manifold is "unusual" if it admits of a continuous self-bijection which is not a homeomorphism. The present paper is a survey of work published over yearsaugmented with recent examples and results.

2000 AMS Classification: 57A05.
Keywords: continuous bijection, 2-manifold.

1. Dissertation.

Signore e signori, ladies and gentlemen!

I am deeply grateful for the opportunity to attend this conference. It is indeed a wonderful occasion, isn't it? The honor to Professor Naimpally is richly deserved, of course. He was once my student, I am very proud to say, and I have basked in the glow of his brilliance for well over thirty five years. In a small way this conference to honor him also honors me and I am happy to accept the reflected glory!

My topic today is far from the theme of this conference but I must say that Professor Naimpally approved of this digression from the main flow. I shall speak about a small twig on a branch of geometric topology which has absorbed me for many years. I trust you will not be disappointed.

For me, it all began in an undergraduate topology course years and years ago. I had just gotten to the point of defining a "homeomorphism" as a continuous bijection with a continuous inverse. An eager young lady in the front row interrupted me. "Why do you have to assume the continuity of the inverse? Couldn't that be proved?" Of course, I had the standard example to give her:

$$f\colon [0,1) \to S^1$$

given by

$$f(t) = (\cos 2\pi t, \sin 2\pi t).$$

This is surely a continuous bijection with a discontinuous inverse. (I almost said "So there!" in triumph.)

The lady was not really happy. "Wasn't that kind of cheat? The two spaces are different. Sure, your example shows that you must assume continuity of f^{-1} in the general case. But what if the map is from a space to itself?" It took a moment's thought to come up with the following example:

Let $X = \{(n, y) : n \in \mathbf{Z}, 0 \leq y < 1\}$. Then a bijection f defined by

$$f(n,y) = \begin{cases} (n,y) & \text{if} \quad n < 0 \\ (n-1,y) & \text{if} \quad n > 1 \\ (0, \frac{1}{2}y) & \text{if} \quad n = 0 \\ (0, \frac{1}{2} + \frac{1}{2}y) & \text{if} \quad n = 1 \end{cases}$$

If you do not "see" this, it simply shrinks two of the "poles", picks one up and stacks it on the other. Well, that kept the lady quiet and I got on with my lecture. She still had another objection, however. She caught up with me in the hall after class and asked for a connected example. I had it ready for her the next day. You may picture this way: At each negative integer point on the real line erect a smal circle tangent to the line. At each of the other integer points stand a unit interval missing its upper endpoint. The bijection simply applies the first map I gave to the first such interval. If you insist on being a purist, you could then shift left one unit. That finally satisfied the student but it left me feeling a little uneasy.

My interest in topology has always been in low-dimensional manifolds. By "low" I mean just two or three dimensions. Anything beyond that is too difficult for me! To make this talk self-contained I define an n-dimensional manifold to be a connected metric space M each point of which has a neighborhood homeomorphic to either $I\!R^n$ or $I\!R^{n-1} \times [0, 1)$. Points of the first kind are *interior points* and make up the *interior* of M, denoted by $IntM$. The second kind are *boundary points* and constitute ∂M, the boundary of M.

While there are only four distinct 1-manifolds, $(0, 1)$, $[0, 1)$, $[0, 1]$, and S^1, the 2-manifolds are much more numerous! Many subsets of $I\!R^2$, including $I\!R^2$ itself, satisfy the definition. The countable infinity of closed orientable surfaces starting with the sphere and going on to the torus and the other spheres with "handles". The corresponding non-orientable surfaces beginning with the projective plane and the Klein bottle constitute the totality of compact boundary-free 2-manifolds. All of these fall into the category I call "usual" manifolds, as you shall soon see.

A minute ago I said I was uneasy about my responses to that nice young lady in my class. I knew she had been recalling the fact that a continuous bijection of the unit interval to itself necessarily has a continuous inverse. I asked myself "Are there manifolds which admit of a continuous bijection with a discontinuous inverse?" Of course, such would have to be non-compact and, of course, I found one quite easily. Think of it as an infinite 2-dimensional tube centered along the x-axis. On it to the left erect an infinity of handles and to the rigth an infinity of pairs of holes with boundaries and chimneys with their top edges missing. The desired bijection simply bends on of the chimneys over until its "missing" upper edge coincides with the boundary of a hole. Draw a picture if you must. But, notice that this is simply my connected example inflated!

As time passed, my colleague, P.H. Doyle, and I developed the topic into several papers. I want to mention some of our results and some of our failures as well (it seems to me that we ought to tell of our errors as often as of our successes. The mathematical community would surely profit from such knowledge!). First however, I must say that Rajagopalan and Wilansky, writing in the Journal of the Australian Mathematical Society in 1966, had introduced the term "non-reversible" to describe a space which admits a continuous bijection which is not a homeomorphism. In our informal talks, however, Doyle and I always spoke of "unusual" spaces and so I now make the following

Definition 1.1. A (Hausdorff) space X is usual if every continuous self-bijection on X is necessarily a homeomorphism: otherwise, X is unusual.

Incidentally, to save a lot of writing and speaking time, from here on a continuous bijection will be simply a *bi-map.*

Theorem 1.2. *For every $n \geq 2$, there are unusual n-manifolds.*

Proof. Look at the example M above and consider $M \times IR^k$, $k \in \mathbf{N}$ □

Any compact manifold is surely usual and, in view of the Brouwer Invariance of Domain Theorem, so is every open manifold. You might say that these facts led us to adopt the term "usual" in the first place. I have to say that with all of the brilliance of Pat Doyle and my own plodding persistence we were never able to characterize either the usual or the unusual manifolds. A first result in that direction was

Theorem 1.3. *If ∂M is compact, then M is usual.*

Let me describe another 2-manifold for your consideration. In IR^2 start with the strip bounded between the x-axis and the line $y = 3$. At all points $(-2n, 2)$ remove an open square of side one centered at $(-2n, 2)$ with sides parallel to the axes. Then remove from the right open rectangles $\{(x, y)$ such that $2k < x < 2k + 1, 1 < y \leq 3\}$. This leaves a series of "towers" to the right. Finally, remove the top edge of each of these towers. Draw this out to see exactly what is entailed.

The bijection is rather obvious, just bend a tower over and let its open end join the edge of a tower to its left. You will notice that there are both compact and non-compact boundary components, infinitely many of each type. DO NOT MAKE FALSE CONJECTURES BASED ON THIS EXAMPLE, AS WE DID AT FIRST! By simply taking the product of this manifold with the unit interval we get an unusual manifold with a connected boundary!

Here is another example to consider: Now I ask you to look at IR^3 and begin with the set of points $\{(x, y, z)$ such that $x^2 + z^2 = 1\}$. From this tube, then, remove open square "patches" centered at the points $(2+4k, \frac{1}{2}, 0)$. From the edge $x = -3$ of the first hole to the left erect a "bridge" over to the right edge $x = -5$ of the hole to the left. Continue this bridge building on to infinity. Finally, to the right at the right edges of the holes erect upright "panels" missing their top edges. The bi-map simply bends one of these panels over to meet the edge of the next hole along. Again, you might want to draw the picture to see just how easy this is. In this example we see that, in the presence of non-compact boundary components, a compact component can be only partially "swallowed" by the action of a bi-map. We wasted a few weeks in a futile search before we came up with this one!

Let me get on to the second of my topics with the following

Definition 1.4. Two (Hausdorff) spaces, X and Y are bijectively related if there are bi-maps $f: X \to Y$ and $g: Y \to X$. We shall use the symbol $Fam(X)$ to denote the equivalence class of all spaces bijectively related to X.

For any usual space $Fam(X) = \{X\}$ but this condition in no way characterizes the usual spaces. I will leave to you to prove that the rationals $\mathbf{Q}$, with the usual topology, have the property $Fam(\mathbf{Q}) = \{\mathbf{Q}\}$.

As you undoubtely expect, there are infinitely many manifolds with non-homeomorphic bijective relatives. I describe next one of the relatives of the second 2-manifold I described earlier. From the edge of the first "hole" to the left remove a closed line segment. Call the manifold so created N. It is easy to see the bi-map from N to M. For the other direction, simply strech the second tower over the first and identify its missing edge with a part of the boundary beyond the first tower. Again, drawing a picture will help considerably in "seeing" this bi-map.

If one draws a simple closed curve J around the hole in N with a closed segment taken away from its edge, one readily proves that M and N are not homeomorphic. Also, it is trivial to see that the class $Fam(M)$ is infinite. We were never able to answer the following question: If $Fam(X)$ is not equal to $\{X\}$, is it necessarily infinite?

It is interesting to note that for the two manifolds M and N above, the products $M \times [0,1)$ and $N \times [0,1)$ are homeomorphic. This led us to another problem we could not solve. Assuming that M and N are non-homeomorphic bijectively related manifolds, when does there exists a manifold P such that $M \times P$ and $N \times P$ are homeomorphic? Can P be compact?

Now suppose that $f\colon M \to N$ and $g\colon N \to M$ make M and N bijectively related. If the composite map $g \circ f\colon M \to M$ is itself a homeomorphism, so is f itself. In the example above, we know that f is not a homeomorphism and therefore M is unusual. This give us

Theorem 1.5. *If $Fam(M)$ is not equal to $\{M\}$, then M is unusual.*

Here are a few more results to give you a flavor of the subject:

Theorem 1.6. *If M and N are bijectively related, then each imbeds in the interior of the other.*

Theorem 1.7. *If M is orientable, so is each of its bijective relatives.*

As an addendum to Theorem 1.7 it is interesting to note that there are bi-maps from orientable manifolds to non-orientable ones but not conversely. I leave you to find an example of a bi-map from the second manifold I described to a non-orientable one. (This is very easy!)

Theorem 1.8. *If $f\colon M \to N$ is a bi-map and if $f(\partial M) = \partial N$, then f is a homeomorphism.*

Theorem 1.9. *If the 2-manifold M has infinitely many compact boundary components, infinitely many handles and infinitely many annular ends, then M is unusual.*

Theorem 1.9 should remind you of my very first example, the genesis of this entire project. Now, let me leave you with problem: Show that for that first manifold M, $Fam(M) = \{M\}$. And here is another puzzle for you to chew on: Consider the 3-manifold consisting of open lower halph-space in $I\!R^3$ together with the open annular disks

$$\{(x,y,0) \text{ such that } \tfrac{1}{4} \le (x-2n)^2 + y^2 \le 1\}.$$

Prove that this is an unusual manifold. If you need a little help with this, consult the K. Whyburn paper listed in the bibliography.

Now I thank you all again and wish you the very best for the future of which you form such an important part!

References

[1] P.H. Doyle and J.G. Hocking, *A decomposition theorem for n-dimensional manifolds*, Proc. Amer. Math. Soc. **13** (1962), 469–471.

[2] P.H. Doyle and J.G. Hocking, *Continuous bijections on manifolds*, J. Austral. Math. Soc. **22** (1976), 257–263.

[3] P.H. Doyle and J.G. Hocking, *Strongly reversible manifolds*, J. Austral. Math. Soc. **Series A** (1983), 172–176.
[4] P.H. Doyle and J.G. Hocking, *Bijectively related spaces I: Manifolds*, Pac. J. Math. **3** No.1 (1984), 23–31.
[5] J. Eichorn, *Die Kompactifizierung offener mannigfaltigjeiten zu geschossenen I*, Math. Nachr. **85** (1978), 5–30.
[6] K. Kuratowski, *Topology Vol 2*, Academic Press, (1968).
[7] D.H. Petty, *One-to-one mappings into the plane*, Fund. Math. **67** (1970), 209–218.
[8] M. Rajagopalan and A. Wilansky, *Reversible topological spaces*, J. Austral. Math. Soc. **6** (1966), 129–138.
[9] K. Whyburn, *A non-topological* $1-1$ *mapping onto* E^3, Bull. Amer. Math. Soc. **71** (1965), 523–537.

Received January 2002
Revised September 2002

John G. Hocking
Department of Mathematics, Michigan State University, East Lansing, Michigan, 48824, USA.

E-mail address: hocking@pilot.msu.edu