APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 16, no. 2(2015), 99-108
doi:10.4995/agt.2015.2988

On cyclic relatively nonexpansive mappings in generalized semimetric spaces

Moosa Gabeleh

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran. (gab.moo@gmail.com, Gabeleh@abru.ac.ir)

Abstract

In this article, we prove a fixed point theorem for cyclic relatively nonexpansive mappings in the setting of generalized semimetric spaces by using a geometric notion of seminormal structure and then we conclude a result in uniformly convex Banach spaces. We also discuss on the stability of seminormal structure in generalized semimetric spaces.

2010 MSC: *47H10; 46B20.*

Keywords: *Cyclic relatively nonexpansive mapping; seminormal structure; generalized semimetric space.*

1. Introduction

A closed convex subset E of a Banach space X has *normal structure* in the sense of Brodskil and Milman ([2]) if for each bounded, closed and convex subset K of E which contains more than one point, there is a point $x \in K$ which is not a diametral point of K, that is, $\sup\{\|x - y\| : y \in K\} < \text{diam}(K)$. In 1965, Kirk proved that if E is a nonempty, weakly compact and convex subset of a Banach space X with normal structure and $T : E \to E$ is a nonexpansive mapping, that is $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in E$, then T has a fixed point ([8]).

As well known, every nonempty, bounded, closed and convex subset of a uniformly convex Banach space X has normal structure. So, the following fixed point theorem concludes from the Kirk's fixed point theorem.

Received 21 May 2014 – Accepted 5 June 2015

Theorem 1.1. *Let E be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space X. Then every nonexpansive mapping $T : E \to E$ has a fixed point.*

Now, let (X, d) be a metric space, and let E, F be subsets of X. A mapping $T : E \cup F \to E \cup F$ is said to be *cyclic* provided that $T(E) \subseteq F$ and $T(F) \subseteq E$. The following interesting theorem is an extension of *Banach contraction principle.*

Theorem 1.2 ([10])**.** *Let E and F be nonempty and closed subsets of a complete metric space (X, d). Suppose that T is a cyclic mapping such that*

$$d(Tx, Ty) \leq \alpha\, d(x, y),$$

for some $\alpha \in (0, 1)$ and for all $x \in E, y \in F$. Then $E \cap F$ is nonempty and T has a unique fixed point in $E \cap F$.

If $E \cap F = \varnothing$ then the cyclic mapping $T : E \cup F \to E \cup F$ cannot have a fixed point, instead it is interesting to study the existence of *best proximity points*, that is, a point $p \in E \cup F$ such that

$$d(p, Tp) = \operatorname{dist}(E, F) := \inf\{d(x, y) : (x, y) \in E \times F\}.$$

Existence of best proximity points for *cyclic relatively nonexpansive* mappings was first studied in [3] (see also [4, 5, 6, 7] for different approaches to the same problem). We recall that the mapping $T : E \cup F \to E \cup F$ is called cyclic relatively nonexpansive provided that T is cyclic on $E \cup F$ and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in E \times F$.

Next theorem was established in [3].

Theorem 1.3 (Corollary 2.1 of [3])**.** *Let E and F be two nonempty, bounded, closed and convex subsets of a uniformly convex Banach space X. Suppose $T : E \cup F \to E \cup F$ is a cyclic relatively nonexpansive mapping. Then T has a best proximity point in $E \cup F$.*

We mention that Theorem 1.3 is based on the fact that every nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X has *proximal normal structure* (see Proposition 2.1 of [3]).

In this article, motivated by Theorem 1.2, we establish a fixed point theorem for cyclic relatively nonexpansive mappings in generalized semimetric spaces. Next we show that if the pair (E, F) considered in Theorem 1.3 has an appropriate geometric condition, then $E \cap F$ must be nonempty and hence, the result follows from Theorem 1.1.

2. Preliminaries

Let X be a set and S a linearly ordered set with its order topology having a smallest element, which denoted by 0. A mapping $D_S : X \times X \to S$ is said to be a *generalized semimetric* provided that for each $x, y \in X$
(1) $D_S(x, y) = 0 \Leftrightarrow x = y$,
(2) $D_S(x, y) = D_S(y, x)$.

If S is the set of nonnegative real numbers, then we replace D_S with D and we say that D is a semimetric on X. Also, if D_S is a generalized semimetric on X, then the pair (X, D_S) is called generalized semimetric space. An easy example of a continuous semimetric which is not a metric is given by letting $X = S = [0,1]$ and defining $D(x,y) := |x-y|^2$ for all $x, y \in X$.

According to Blumenthal ([1]; p.10), D_S generates a topology on X as follows:
A point $p \in X$ is said to be a limit point of a subset E of X if given any $\alpha \in S$ with $\alpha \neq 0$, there exists a point $q \in E$ such that $D_S(p,q) \in (0,\alpha) := \{\beta \in S : 0 < \beta < \alpha\}$. A set E in X is said to be closed if it contains all of its limit points and a set U in X is said to be open if $X - U$ is closed. If D_S is a continuous mapping w.r.t. the topology on X induced by D_S, then D_S is said to be a *continuous generalized semimetric*.

Given a generalized semimetric D_S, a B-set will be a set like

$$\mathcal{B}(x;\alpha) := \{u \in X : D_S(x,u) \leq \alpha\}.$$

We say that a set $E \subseteq X$ is *spherically bounded* if there exists a B-set which contains E. We also define

$$\operatorname{cov}(E) := \bigcap\{K : K \text{ is a } B\text{-set containing } E\}.$$

Definition 2.1. A subset E of a generalized semimetric space (X, D_S) is said to be admissible if $E = \operatorname{cov}(E)$.

The collection of all admissible subsets of a generalized semimetric (X, D_S) will be denoted by $\mathcal{A}(X)$. We will say that $\mathcal{A}(X)$ is *compact* provided that any descending chain of nonempty members of $\mathcal{A}(X)$ has nonempty intersection.

The linearly ordered set S is said to have *least upper bound property* (lub-property) if each set in S which is bounded above has a smallest upper bound. Dually, this implies that S has the *greatest lower bound property* (glb- property). We mention that if S is *connected* relative to its order topology, then S has the lub- property.

Let (E, F) be a nonempty pair of subsets of a generalized semimetric (X, D_S). We shall adopt the following notations.

$$\begin{aligned}
\operatorname{dist}(E,F) &:= \text{glb } \{D_S(x,y) : (x,y) \in E \times F\},\\
\delta_x(E) &:= \text{lub } \{D_S(x,u) : u \in E\},\ \forall x \in X,\\
\delta(E,F) &:= \text{lub } \{\delta_x(F) : x \in E\},\\
\operatorname{diam}(E) &:= \delta(E,E).\\
E_0 &:= \{x \in E : D_S(x,y) = dist(A,B),\ \text{for some } y \in B\},\\
F_0 &:= \{y \in F : D_S(x,y) = dist(A,B),\ \text{for some } x \in A\}.
\end{aligned}$$

Definition 2.2 ([6]). A pair of sets (E, F) in a generalized semimetric space (X, D_S) is said to be a proximal compactness pair provided that every net $\{(x_\alpha, y_\alpha)\}$ of $E \times F$ satisfying the condition that $D_S(x_\alpha, y_\alpha) \to \operatorname{dist}(E,F)$, has a convergent subnet in $E \times F$.

3. Seminormal Structure

Throughout this paper, we shall say that a pair (E, F) of subsets of a generalized semimetric space (X, D_S) satisfies a property if both E and F satisfy that property. For example, (E, F) is admissible if and only if both E and F are admissible; $(E, F) \subseteq (G, H) \Leftrightarrow E \subseteq G,$ and $F \subseteq H$.

Let (E, F) be a nonempty pair of admissible subsets of X. We say that the pair (E, F) satisfies the condition (P) if E contained in a B-set centered at a point of F and the set F contained in a B-set centered at a point of E. Also, for the pair (E, F) we define

$$\mathcal{R}(E) := \{\alpha \in S : [\bigcap_{y\in F} \mathcal{B}(y;\alpha)] \cap E \neq \varnothing\},$$

$$\mathcal{R}(F) := \{\beta \in S : [\bigcap_{x\in E} \mathcal{B}(x;\beta)] \cap F \neq \varnothing\}.$$

Note that the if the pair (E, F) satisfies the condition (P), then $(\mathcal{R}(E), \mathcal{R}(F))$ is a nonempty pair of subsets of S. Indeed, if $E \subseteq \mathcal{B}(v;\beta)$ for some $v \in F$ and $\beta \in S$, then $D_S(x, v) \leq \beta$ for all $x \in E$ and so, $v \in \mathcal{B}(x;\beta)$ for all $x \in E$. Thus $v \in \bigcap_{x\in E} \mathcal{B}(x;\beta) \cap F$ i.e. $\beta \in \mathcal{R}(F)$. Similarly, we can see that $\mathcal{R}(E)$ is nonempty.

Furthermore, we set

$$r(E) := \text{glb } \mathcal{R}(E),\ r(F) := \text{glb } \mathcal{R}(F) \text{ and } \rho := \text{lub } \{r(E), r(F)\},$$

and define

$$\mathcal{C}_F(E) := \{x \in E : x \in \bigcap_{y\in F} \mathcal{B}(y;\rho)\},$$

$$\mathcal{C}_E(F) := \{y \in F : y \in \bigcap_{x\in E} \mathcal{B}(x;\rho)\}.$$

Next lemma guarantees that $(\mathcal{C}_F(E), \mathcal{C}_E(F))$ is a nonempty pair.

Lemma 3.1. *Let (X, D_S) be a generalized semimetric space such that $\mathcal{A}(X)$ is compact and S is connected. Let (E, F) be a nonempty and admissible pair of subsets of X such that (E, F) satisfies the condition (P). Then $(\mathcal{C}_F(E), \mathcal{C}_E(F))$ is a nonempty and admissible pair in X which satisfies the condition (P).*

Proof. Let $\alpha > \rho$ and $\beta > \rho$ be such that the pair $(\mathcal{C}_\alpha(E), \mathcal{C}_\beta(F))$ is nonempty, where

$$\mathcal{C}_\alpha(E) := [\bigcap_{y\in F} \mathcal{B}(y;\alpha)] \cap E \quad \& \quad \mathcal{C}_\beta(F) := [\bigcap_{x\in E} \mathcal{B}(x;\beta)] \cap F.$$

We show that $\mathcal{C}_F(E) = \bigcap_{\alpha\geq\rho} \mathcal{C}_\alpha(E)$ and $\mathcal{C}_E(F) = \bigcap_{\beta\geq\rho} \mathcal{C}_\beta(F)$. Suppose that $u \in \bigcap_{\alpha\geq\rho} \mathcal{C}_\alpha(E)$. If u is not member of $\mathcal{C}_F(E)$, then there exists $v \in F$ such that $D_S(u, v) > \rho$. Since S is connected, there exists an element $\gamma \in S$ such that $\rho < \gamma < D_S(u, v)$. But this is a contradiction by the fact that $u \in \mathcal{C}_\gamma(E)$. That is, $u \in \mathcal{C}_F(E)$ and so, $\bigcap_{\alpha\geq\rho}\mathcal{C}_\alpha(E) \subseteq \mathcal{C}_F(E)$. This implies that $\mathcal{C}_F(E) \neq \varnothing$. Besides, if $u \in \mathcal{C}_F(E)$, then

$$u \in [\bigcap_{y\in F} \mathcal{B}(y;\rho)] \cap E \subseteq [\bigcap_{y\in F} \mathcal{B}(y;\alpha)] \cap E = \mathcal{C}_\alpha(E),\ \forall \alpha \geq \rho.$$

Hence, $u \in \bigcap_{\alpha\geq\rho}\mathcal{C}_\alpha(E)$ which deduces that $\mathcal{C}_F(E) = \bigcap_{\alpha\geq\rho}\mathcal{C}_\alpha(E)$. Similar argument implies that $\mathcal{C}_E(F) = \bigcap_{\beta\geq\rho}\mathcal{C}_\alpha(F)$. Now, suppose that $E \subseteq \mathcal{B}(q,\gamma_1)$ and $F \subseteq \mathcal{B}(p,\gamma_2)$ for some $(p,q) \in E \times F$ and $\gamma_1,\gamma_2 \in S$. Put $\gamma := \text{lub}\,\{\gamma_1,\gamma_2\}$. Then for each $\alpha \in S$ with $\alpha \geq \rho$, we have $\mathcal{C}_\alpha(E) \subseteq \mathcal{B}(q,\gamma)$ which concludes that

$$\mathcal{C}_F(E) = \bigcap_{\alpha\geq\rho} \mathcal{C}_\alpha(E) \subseteq \mathcal{B}(q,\gamma).$$

Similar argument implies that $\mathcal{C}_E(F) \subseteq \mathcal{B}(p,\gamma)$. That is, the pair $(\mathcal{C}_F(E), \mathcal{C}_E(F))$ satisfies the condition (P). □

Let (E,F) be a nonempty and admissible pair of subsets of a generalized semimetric space (X,D_S) such that (E,F) satisfies the condition (P). In what follows we set

$$\Sigma_{(E,F)} := \{(G,H) \subseteq (E,F) : G,H \in \mathcal{A}(X) \text{ and } (G,H) \text{ satisfies the condition (P)}\}.$$

Here, we introduce the following geometric notion on a nonempty and admissible pair in generalized semimetric spaces.

Definition 3.2. Suppose that (E,F) is a nonempty and admissible pair of subsets of a generalized semimetric space (X,D_S) such that (E,F) satisfies the condition (P) and $\mathcal{A}(X)$ is compact. We say that $\Sigma_{(E,F)}$ has seminormal structure if for each $(G,H) \in \Sigma_{(E,F)}$, either $G \cup H$ is singleton or $\mathcal{C}_H(G) \subsetneq G$, $\mathcal{C}_G(H) \subsetneq H$.

We now state the main result of this paper.

Theorem 3.3. *Let (X,D_S) be a generalized semimetric space, where S is connected w.r.t. its order topology and let $\mathcal{A}(X)$ be compact. Suppose that (E,F) is a nonempty and admissible pair of subsets of X which satisfies the condition (P) and $\Sigma_{(E,F)}$ has seminormal structure. If $T : E\cup F \to E \cup F$ is a cyclic relatively nonexpansive mapping, then $E \cap F$ is nonempty and T has a fixed point in $E \cap F$.*

Proof. Put

$$\mathcal{F} := \{(G,H) : (G,H) \in \Sigma_{(E,F)} \text{ and } T \text{ is cyclic on } G \cup H\}.$$

By the fact that $\mathcal{A}(X)$ is compact and by using Zorn's lemma, we conclude that $\mathcal{F}$ has a minimal element say $(K_1,K_2) \in \mathcal{F}$. Since $T(K_1) \subseteq K_2$ and $K_2 \in \mathcal{A}(X)$, we deduce that $\text{cov}(T(K_1)) \subseteq K_2$. Then

$$T(\text{cov}(T(K_1))) \subseteq T(K_2) \subseteq \text{cov}(T(K_2)).$$

Similarly, we can see that $T(\text{cov}(T(K_2))) \subseteq \text{cov}(T(K_1))$, that is, T is cyclic on $\text{cov}(T(K_2)) \cup \text{cov}(T(K_1))$. Besides, $(\text{cov}(T(K_2)), \text{cov}(T(K_1)))$ satisfies the condition (P). Indeed, if $K_1 \subseteq \mathcal{B}(q, \alpha)$ for some $q \in K_2$ and $\alpha \in S$, then for each $x \in K_1$, we have

$$D_S(Tx, Tq) \leq D_S(x, q) \leq \alpha,$$

that is, $Tx \in \mathcal{B}(Tq, \alpha)$ for each $x \in K_1$. So, $T(K_1) \subseteq \mathcal{B}(Tq, \alpha)$. Thus $\mathrm{cov}(T(K_1)) \subseteq \mathcal{B}(Tq, \alpha)$. Similarly, if $K_2 \subseteq \mathcal{B}(p, \beta)$ for some $p \in K_1$ and $\beta \in S$, then we can see that $\mathrm{cov}(T(K_2)) \subseteq \mathcal{B}(Tp, \beta)$. Hence, $(\mathrm{cov}(T(K_2)), \mathrm{cov}(T(K_1)))$ satisfies the condition (P). Minimality of (K_1, K_2) implies that

$$K_1 = \mathrm{cov}(T(K_2)) \ \& \ K_2 = \mathrm{cov}(T(K_1)).$$

It follows from Lemma 3.1 that $(\mathcal{C}_{K_2}(K_1), \mathcal{C}_{K_1}(K_2))$ is a nonempty member of $\Sigma_{(E,F)}$. We show that T is cyclic on $\mathcal{C}_{K_2}(K_1) \cup \mathcal{C}_{K_1}(K_2)$. Let $x \in \mathcal{C}_{K_2}(K_1)$. Then $x \in [\bigcap_{y \in K_2} \mathcal{B}(y; \rho)] \cap K_1$. So, $D_S(x, y) \leq \rho$ for each $y \in K_2$. Since T is cyclic relatively nonexpansive,

$$D_S(Tx, Ty) \leq D_S(x, y) \leq \rho, \ \forall y \in K_2.$$

Thus $T(K_2) \subseteq \mathcal{B}(Tx; \rho)$ which implies that

$$K_1 = \mathrm{cov}(T(K_2)) \subseteq \mathcal{B}(Tx; \rho).$$

Hence, $Tx \in [\bigcap_{u \in K_1} \mathcal{B}(u; \rho)] \cap K_2 = \mathcal{C}_{K_1}(K_2)$. That is, $T(\mathcal{C}_{K_2}(K_1)) \subseteq \mathcal{C}_{K_1}(K_2)$. Similarly, we can see that $T(\mathcal{C}_{K_1}(K_2)) \subseteq \mathcal{C}_{K_2}(K_1)$. Thereby, T is cyclic on $\mathcal{C}_{K_2}(K_1) \cup \mathcal{C}_{K_1}(K_2)$. So, $(\mathcal{C}_{K_2}(K_1), \mathcal{C}_{K_1}(K_2)) \in \mathcal{F}$. Again, by the minimality of (K_1, K_2) we must have

$$\mathcal{C}_{K_2}(K_1) = K_1 \quad \& \quad \mathcal{C}_{K_1}(K_2) = K_2.$$

Since $\Sigma_{(E,F)}$ has the seminormal structure, we deduce $K_1 = K_2 = \{p\}$ for some $p \in X$. Therefore, $p \in E \cap F$ is a fixed point of T.

□

Remark 3.4. Note that in Theorem 3.3 we have not the assumption of continuity of D_S. We also mention that if the mapping T considered in Theorem 3.3 is nonexpansive self-mapping, the the main result of [9] is deduces (see Theorem 3 of [9] for more information).

Definition 3.5. Let (E, F) be a nonempty and admissible pair of subsets of a semimetric space (X, D) such that (E, F) satisfies the condition (P). We say that (E, F) has the property UC if for each nonempty pair $(G, H) \in \Sigma_{(E,F)}$ and for any $\varepsilon > 0$, there exists $\alpha(\varepsilon) > 0$ such that for all $R > 0$ and $x_1, x_2 \in G$ and $y \in H$ with

$$D(x_1, y) \leq R, \quad D(x_2, y) \leq R \ \text{ and } \ D(x_1, x_2) \geq R\varepsilon,$$

there exists $u \in G$ such that $D(u, y) \leq R(1 - \alpha(\varepsilon)) < R$.

We now prove the following existence theorem.

Theorem 3.6. *Let (X, D) be a semimetric space such that D is continuous and $\mathcal{A}(X)$ is compact. Suppose (E, F) is a nonempty and admissible pair such that $E_0 \neq \varnothing$ and (E, F) satisfies the condition (P). Assume that (E, F) is a proximal compactness pair which has the property UC. If $T : E \cup F \to E \cup F$ is* *a cyclic relatively nonexpansive mapping, then either $E \cap F$ is nonempty and T has a fixed point in $E \cap F$, or T has a best proximity point in $E \cup F$.*

Proof. Let

$$\mathcal{F}' := \{(G,H) \in \Sigma_{(E,F)} \text{ s.t. } \exists (x,y) \in G \times H \text{ with } D(x,y) = dist(E,F) \text{ and } T \text{ is cyclic on } G \cup H\}.$$

Since $E_0 \neq \varnothing$, $(E,F) \in \mathcal{F}'$. Moreover, if (G_α, H_α) is a descending chain in $\mathcal{F}'$ and put $G := \bigcap_\alpha G_\alpha$ and we set $H := \bigcap_\alpha H_\alpha$, then by the compactness of $\mathcal{A}(X)$, (G,H) is a nonempty member of $\Sigma_{(E,F)}$ and obviously, T is cyclic on $G \cup H$. Now, suppose for each α there exists $(x_\alpha, y_\alpha) \in G_\alpha \times H_\alpha$ such that $D(x_\alpha, y_\alpha) = \text{dist}(E,F)$. Since (E,F) is proximal compactness, $\{(x_\alpha, y_\alpha)\}$ has a convergent subnet say $\{(x_{\alpha_i}, y_{\alpha_i})\}$ such that $x_{\alpha_i} \to x \in E$ and $y_{\alpha_i} \to y \in F$. Hence,

$$D(x,y) = \lim_i D(x_{\alpha_i}, y_{\alpha_i}) = \text{dist}(E,F),$$

that is, there exists an element $(x,y) \in G \times H$ such that $D(x,y) = \text{dist}(E,F)$. So, every increasing chain in $\mathcal{F}'$ is bounded above with respect to revers inclusion relation. Using Zorn's lemma, we obtain a minimal element for $\mathcal{F}'$, say (K_1, K_2). If $K_1 \cup K_2$ is singleton, then T has a fixed point in $E \cap F$ and we are finished. So, we assume that $K_1 \cup K_2$ is not singleton. Similar argument of Theorem 3.3 concludes that $\mathcal{C}_{K_2}(K_1) = K_1$ and $\mathcal{C}_{K_1}(K_2) = K_2$. We now consider the following :

Case 1. If $\min\{\text{diam}(K_1), \text{diam}(K_2)\} = 0$.

We may assume that $K_1 = \{p\}$ for some element $p \in E$. Let $q \in K_2$ be such that $D(p,q) = \text{dist}(E,F)$. Since T is cyclic relatively nonexpansive mapping,

$$D(Tp, p) = D(Tp, Tq) \leq D(p,q) = \text{dist}(E,F),$$

that is, p is a best proximity point of T and the result follows.

Case 2. If $\min\{\text{diam}(K_1), \text{diam}(K_2)\} > 0$.

Put

$$R := \delta(K_1, K_2) \text{ and } r := \min\{\text{diam}(K_1), \text{diam}(K_2)\}.$$

Let $x_1, x_2 \in K_1$ be such that $D(x_1, x_2) \geq \frac{1}{2}\text{diam}(K_1)$ and let $\varepsilon > 0$ be such that $R\varepsilon \leq \frac{r}{2}$. Now, for each $y \in K_2$ we have

$$D(x_1, y) \leq R, \;\; D(x_2, y) \leq R \text{ and } D(x_1, x_2) \geq \frac{1}{2} r \geq R\varepsilon.$$

Since (E,F) has the property UC, there exists $\alpha(\varepsilon) > 0$ and $u \in K_1$ so that

$$D(u, y) \leq R(1 - \alpha(\varepsilon)), \;\; \forall y \in K_2.$$

Then $u \in [\bigcap_{y \in K_2} \mathcal{B}(y; R(1-\alpha(\varepsilon)))] \cap K_1$, that is, $[\bigcap_{y \in K_2} \mathcal{B}(y; R(1-\alpha(\varepsilon)))] \cap K_1 \neq \varnothing$. Similarly, we can see that $[\bigcap_{x \in K_1} \mathcal{B}(x; R(1-\alpha(\varepsilon)))] \cap K_2 \neq \varnothing$. Set

$$r(K_1) := \inf\{s > 0 : [\bigcap_{y \in K_2} \mathcal{B}(y; s)] \cap K_1 \neq \varnothing\},$$

$$r(K_2) := \inf\{s > 0 : [\bigcap_{x \in K_1} \mathcal{B}(x; s)] \cap K_2 \neq \varnothing\}.$$

Note that for $\rho := \max\{r(K_1), r(K_2)\}$ we have $\rho \le R(1-\alpha(\varepsilon))$. Since $\mathcal{C}_{K_2}(K_1) = K_1$,

$$x \in \bigcap_{y\in K_2} \mathcal{B}(y;\rho),\ \forall x \in K_1,$$

which implies that $\delta_x(K_2) \le \rho$ for all $x \in K_1$. Thus

$$R = \delta(K_1, K_2) = \sup_{x\in K_1} \delta_x(K_2) \le \rho \le R(1-\alpha(\varepsilon)) < R,$$

which is a contradiction and this completes the proof of Theorem. □

Next corollary is a straightforward consequence of Theorem 3.6 in the setting of uniformly convex Banach spaces.

Corollary 3.7 (see [3]). *Suppose that (E, F) is a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X. Let $T : E\cup F \to E\cup F$ be a cyclic relatively nonexpansive mapping. Then either $E\cap F$ is nonempty and T has a fixed point in $E \cap F$ or T has a best proximity point in $E \cup F$.*

Example 3.8. Let $X = \mathbb{R}$ and let $A := [-1, 1]$. Define the mapping $T : A \to A$ with

$$T(x) = \begin{cases} -x & \text{if } x \in [-1,0], \\ -x & \text{if } x \in [0,1]\cap\mathbb{Q}, \\ 0 & \text{if } x \in [0,1]\cap\mathbb{Q}^c. \end{cases}$$

Then T is a self-mapping defined on a nonempty bounded, closed and convex subset of X. Note that existence of fixed point of T cannot be deduced from Theorem 1.1, because of the fact T is not continuous (and so is not nonexpansive). Now, Suppose $E := [-1, 0]$ and $F := [0, 1]$ and formulate the mapping $T : E \cup F \to E \cup F$ as follows:

$$T(x) = \begin{cases} -x & \text{if } x \in E, \\ -x & \text{if } x \in F\cap\mathbb{Q}, \\ 0 & \text{if } x \in F\cap\mathbb{Q}^c. \end{cases}$$

It is easy to see that $\|Tx - Ty\| \le \|x - y\|$ for all $(x, y) \in E \times F$, that is, T is cyclic relatively nonexpansive mapping on the nonempty, bounded, closed and convex pair (E, F). Hence, the existence of fixed point for T is concluded from Corollary 3.7.

4. Stability and Seminormal Structure

We begin our main conclusions of this section with the following notion.

Definition 4.1. Let (X, D_S) be a generalized semimetric space and let (E, F) be a nonempty pair of subsets of X. A mapping $T : E \cup F \to E \cup F$ is said to be cyclic relatively h-nonexpansive for some $h \in S$ with $h > 0$ if

$$D_S(Tx, Ty) \le \text{lub}\ \{D_S(x, y), h\},$$

for all $(x, y) \in E \times F$.

Here, we state the following stability result for cyclic relatively h-nonexpansive mappings.

Theorem 4.2. *Let (X, D_S) be a generalized semimetric space, where S is connected w.r.t. its order topology and let $\mathcal{A}(X)$ be compact. Suppose that (E, F) is a nonempty and admissible pair of subsets of X which satisfies the condition (P) and $\Sigma_{(E,F)}$ has seminormal structure. If $T : E \cup F \to E \cup F$ is a cyclic relatively h-nonexpansive mapping, then there exists an element $p \in A \cup B$ so that $D_S(p, Tp) \le h$.*

Proof. Similar argument of Theorem 3.3 implies that there exists a nonempty and admissible pair of subsets $(K_1, K_2) \subseteq (E, F)$ which satisfies the condition (P) and by minimality,

$$\mathrm{cov}(T(K_2)) = K_1 \quad \text{and} \quad \mathrm{cov}(T(K_1)) = K_2.$$

If $K_1 \cup K_2$ is singleton, the result follows. So, assume that $\mathcal{C}_{K_2}(K_1) \subsetneqq K_1$ and $\mathcal{C}_{K_1}(K_2) \subsetneqq K_2$. Let u be an arbitrary element of $\mathcal{C}_{K_2}(K_1)$. Suppose $\rho < h$. Then $D_S(u, y) \le \rho$ for all $y \in K_2$. Since T is cyclic on $K_1 \cup K_2$, we have $D_S(u, Tu) \le \rho < h$ and we are finished. We now suppose that $h \le \rho$. Let $y \in K_2$. If $D_S(u, y) \ge h$, then

$$D_S(Tu, Ty) \le \ \mathrm{lub}\{D_S(u, y), h\} = D_S(u, y) \le \rho.$$

Besides, if $D_S(u, y) < h$, then

$$D_S(Tu, Ty) \le \ \mathrm{lub}\{D_S(u, y), h\} = h \le \rho,$$

that is, for each $y \in K_2$ we have $D_S(Tu, Ty) \le \rho$ which implies that $Ty \in \mathcal{B}(Tu; \rho)$ for all $y \in K_2$. Hence, $T(K_2) \subseteq \mathcal{B}(Tu; \rho)$. So,

$$K_1 = \mathrm{cov}(T(K_2)) \subseteq \mathcal{B}(Tu; \rho),$$

and then $Tu \in [\bigcap_{x \in K_1} \mathcal{B}(x; \rho)] \cap K_2$. Thus $Tu \in \mathcal{C}_{K_1}(K_2)$. Thereby, $T(\mathcal{C}_{K_2}(K_1)) \subseteq \mathcal{C}_{K_1}(K_2)$. By a similar argument we obtain $T(\mathcal{C}_{K_1}(K_2)) \subseteq \mathcal{C}_{K_2}(K_1)$. Therefore, T is cyclic on $T(\mathcal{C}_{K_2}(K_1)) \cup \mathcal{C}_{K_1}(K_2)$. Minimality of (K_1, K_2) deduces that

$$K_1 = \mathcal{C}_{K_2}(K_1) \text{ and } K_2 = \mathcal{C}_{K_1}(K_2),$$

which is a contradiction. □

Acknowledgements. *This research was in part supported by a grant from IPM (No. 93470047).*

References

[1] L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford Univ. Press, London (1953).

[2] M. S. Brodskii and D. P. Milman, *On the center of a convex set*, Dokl. Akad. Nauk. **USSR 59** (1948), 837–840 (in Russian).

[3] A. A. Eldred, W. A. Kirk and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math. **171** (2005), 283–293.

[4] R. Espínola, *A new approach to relatively nonexpansive mappings*, Proc. Amer. Math. Soc. **136** (2008), 1987–1996.

[5] R. Espínola and M. Gabeleh, *On the structure of minimal sets of relatively nonexpansive mappings*, Numer. Funct. Anal. Optim. **34** (2013), 845–860.

[6] M. Gabeleh, *Minimal sets of noncyclic relatively nonexpansive mappings in convex metric spaces*, Fixed Point Theory, to appear.

[7] M. Gabeleh and N. Shahzad, *Seminormal structure and fixed points of cyclic relatively nonexpansive mappings* , Abstract Appl. Anal. **2014** (2014), Article ID 123613, 8 pages.

[8] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

[9] W. A. Kirk and B. G. Kang *A fixed point theorem revisited*, J. Korean Math. Soc. **34** (1997), 285–291.

[10] W. A. Kirk, P. S. Srinivasan amd P. Veeramani,*Fixed points for mappings satisfying cyclic contractive conditions*, Fixed Point Theory **4**, no. 1 (2003), 79–86.