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Topological n-cells and Hilbert cubes in inverse limits

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Abstract

It has been shown by S. Mardešić that if a compact metrizable space X has $\dim X \geq 1$ and X is the inverse limit of an inverse sequence of compact triangulated polyhedra with simplicial bonding maps, then X must contain an arc. We are going to prove that if $\mathbf{X} = (|K_a|, p^b_a, (A, \preceq))$ is an inverse system in set theory of triangulated polyhedra $|K_a|$ with simplicial bonding functions p^b_a and $X = \lim \mathbf{X}$, then there exists a uniquely determined sub-inverse system $\mathbf{X}_X = (|L_a|, p^b_a||L_b|, (A, \preceq))$ of $\mathbf{X}$ where for each a, L_a is a subcomplex of K_a, each $p^b_a||L_b| : |L_b| \to |L_a|$ is surjective, and $\lim \mathbf{X}_X = X$. We shall use this to generalize the Mardešić result by characterizing when the inverse limit of an inverse sequence of triangulated polyhedra with simplicial bonding maps must contain a topological n-cell and do the same in the case of an inverse system of finite triangulated polyhedra with simplicial bonding maps. We shall also characterize when the inverse limit of an inverse sequence of triangulated polyhedra with simplicial bonding maps must contain an embedded copy of the Hilbert cube. In each of the above settings, all the polyhedra have the weak topology or all have the metric topology (these topologies being identical when the polyhedra are finite).

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1. Introduction

Theorem 4.10.10 of [10] reads as follows.

Theorem 1.1. *Every completely metrizable space X is homeomorphic to the inverse limit of an inverse sequence $(|K_i|_m, p_i^{i+1})$ of metric polyhedra and* PL *maps such that each K_i is locally finite-dimensional,* $\operatorname{card} K_i \leq \operatorname{wt} X$*, and each bonding map $p_i^{i+1} : |K_{i+1}|_m \to |K_i|_m$ is simplicial for some admissible subdivision K_i' of K_i, where admissibility guarantees the continuity of $p_i^{i+1} : |K_{i+1}|_m \to |K_i|_m$.*

The notion of locally finite-dimensional used in Theorem 1.1 goes this way. Let K be a simplicial complex. Whenever v is a vertex of K, then $\overline{\mathrm{st}}(v, K)$ will be the closed star of v in K, which is the subcomplex of K consisting of the simplexes of K having v as a vertex and all faces of such simplexes. Then K is called *locally finite-dimensional* if $\dim(\overline{\mathrm{st}}(v, K)) < \infty$ for each $v \in K^{(0)}$.

One might wonder if an inverse sequence such as that in Theorem 1.1 could be designed so that all the bonding maps[1] are simplicial with respect to the given triangulations; unfortunately this is not the case. It was shown by S. Mardešić in Theorem 2.1 of [7], that if a compact metrizable space X has $\dim X \geq 1$ and X is the inverse limit of an inverse sequence of compact triangulated polyhedra with simplicial bonding maps, then X must contain an arc. Since pseudo-arcs (see [8]) are metrizable compacta with $\dim \geq 1$ that contain no arcs, then he was able to obtain Corollary 2.2 of [7], which says that there exist metrizable compacta that cannot be written as the limit of an inverse sequence of compact triangulated polyhedra with simplicial bonding maps. The proof of Theorem 2.1 of [7] is given without the assumption that the bonding maps are surjective, but if they were, then by an observation of M. Levin, its proof would be trivial.

The question of whether a given metrizable compactum could be written as the limit of an inverse sequence of compact triangulated polyhedra with simplicial bonding maps arose from our research in [9]. There we were able to find, for the sake of extension theory, a "substitute" Z for any given compact metrizable space X. This metrizable compactum Z is represented as the limit of an inverse sequence of finite triangulated polyhedra in such a manner that all the bonding maps are simplicial with respect to these triangulations. Since the process of determining such a Z was complex, we were concerned to know if it was necessary, that is, could we represent the given X "simplicially" from the outset; the result of [7] made it apparent that we could not escape such a complication.

We shall demonstrate, Proposition 2.7, that if $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ is an inverse system in set theory of triangulated polyhedra $|K_a|$ with simplicial bonding functions p_a^b, and $X = \lim \mathbf{X}$, then there exists a uniquely determined sub-inverse system $\mathbf{X}_X = (|L_a|, p_a^b||L_b|, (A, \preceq))$ of $\mathbf{X}$ where for each a, L_a is a subcomplex of K_a, each $p_a^b||L_b| : |L_b| \to |L_a|$ is surjective, and $\lim \mathbf{X}_X = X$.

[1] In this paper *map* means continuous function.

Hence for such a "simplicial" inverse system in which the polyhedra $|K_a|$ are given either the CW (weak) topology or the metric topology m, one may as well assume for topological purposes that the bonding functions are surjective maps.

In Corollary 3.3 we will characterize when the limit of an inverse sequence of triangulated polyhedra with simplicial bonding maps must contain a topological n-cell. In Proposition 3.5, we display a similar characterization in case we are dealing with an inverse system of finite polyhedra and simplicial bonding maps. Our Theorem 4.13 characterizes when the limit of an inverse sequence of triangulated polyhedra and simplicial bonding maps must contain a copy of the Hilbert cube I^∞. We were not successful in obtaining such a result for inverse systems even in the case that the coordinate spaces are finite polyhedra. In Section 5 we shall provide what we could do for such systems.

2. Simplicial Inverse Systems

Let K be a simplicial complex. Then by $|K|_{\mathrm{CW}}$ we mean the polyhedron $|K|$ with the CW-topology (sometimes called the weak topology) and by $|K|_m$ we mean $|K|$ with the metric topology m.[2] If K is finite, then the CW-topology is the same as the metric topology m, so we usually just write $|K|$ with no subscript. In case L is a simplicial complex and $f : K \to L$ is a simplicial function, then f induces a function $|f| : |K| \to |L|$ which we say is simplicial from $|K|$ to $|L|$. In this setting we usually just write f instead of $|f|$; moreover, one has that both $f : |K|_{\mathrm{CW}} \to |L|_{\mathrm{CW}}$ and $f : |K|_m \to |L|_m$ are maps.

We shall be concerned with inverse systems $\mathbf{X} = (X_a, p^b_a, (A, \preceq))$ with a directed set $(A, \preceq)$ as indexing set. If $X = \lim \mathbf{X}$, then $p_a : X \to X_a$ will denote the a-coordinate projection. For $x \in X$, we shall typically write $p_a(x) = x_a$, and denote $x = (x_a)_{a\in A}$ or just $x = (x_a)$. If for each $a \in A$, $Y_a \subset X_a$ and whenever $a \preceq b$, $p^b_a(Y_b) \subset Y_a$, then we call $\mathbf{Y} = (Y_a, p^b_a|Y_b, (A, \preceq))$ a *sub-inverse system* of $\mathbf{X}$. Clearly $\lim \mathbf{Y} \subset \lim \mathbf{X}$. In case $(A, \preceq)$ is $(\mathbb{N}, \leq)$, we simply denote the inverse system $\mathbf{X} = (X_i, p^{i+1}_i)$ and call it an *inverse sequence*.

The main result of this section is Proposition 2.7. It shows that if X is the inverse limit of an inverse system in set theory of triangulated polyhedra and simplicial maps, then there is a sub-inverse system consisting of subpolyhedra determined by subcomplexes of the given triangulations such that the limit of this sub-inverse system is X and that the restricted, and hence simplicial, maps are surjective.

Definition 2.1. Let $\mathbf{X} = (|K_a|, p^b_a, (A, \preceq))$ be an inverse system in set theory of triangulated polyhedra and simplicial bonding functions p^b_a. We shall refer to $\mathbf{X}$ as a **simplicial inverse system**. In case all $|K_a|$ have the topology CW or all have the topology m, then we shall denote all $|K_a|$ respectively as $|K_a|_{\mathrm{CW}}$ or $|K_a|_m$, and understand that all the functions p^b_a in set theory are simultaneously maps. If all the functions p^b_a are surjective, then we shall call

[2] One may consult [10] for more information about polyhedra.

$\mathbf{X}$ a **surjective** inverse system. Let $X = \lim \mathbf{X}$, $x \in X$, and for each $a \in A$, denote by $\sigma_{x,a}$ the unique simplex of K_a with $x_a \in \operatorname{int} \sigma_{x,a}$.

Lemma 2.2. *Let $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ be a simplicial inverse system, $X = \lim \mathbf{X}$, and $x \in X$. Then the trace $\{\sigma_{x,a} \,|\, a \in A\}$ of x in $\mathbf{X}$ has the property that whenever $a \preceq b$, $p_a^b(\sigma_{x,b}) = \sigma_{x,a}$. Hence $\mathbf{X}_x = (\sigma_{x,a}, p_a^b|\sigma_{x,b}, (A, \preceq))$ is a surjective simplicial sub-inverse system of $\mathbf{X}$ with bonding functions that are simultaneously maps. Moreover, $x \in \lim \mathbf{X}_x \subset X$.*

Definition 2.3. We shall refer to the uniquely determined inverse system $\mathbf{X}_x = (\sigma_{x,a}, p_a^b|\sigma_{x,b}, (A, \preceq))$ of Lemma 2.2 as the **trace** of x in $\mathbf{X}$.

Definition 2.4. Let $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ be a simplicial inverse system, $X = \lim \mathbf{X}$, $Q \subset X$, for each $a \in A$ denote $M_{Q,a} = \{\sigma_{y,a} \,|\, y \in Q\}$, and define $L_{Q,a}$ to be the collection of faces of elements of $M_{Q,a}$.

Lemma 2.5. *Let $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ be a simplicial inverse system, $X = \lim \mathbf{X}$, and $Q \subset X$. Then for each $a \in A$:*

1. *$L_{Q,a}$ is a uniquely determined subcomplex of K_a,*
2. *if $n \in \mathbb{N}$, and for all $y \in Q$, $\dim \sigma_{y,a} \leq n$, then $\dim L_{Q,a} \leq n$, and*
3. *if $b \in A$ and $a \preceq b$, $p_a^b(|L_{Q,b}|) = |L_{Q,a}|$.*

Hence $\mathbf{X}_Q = (|L_{Q,a}|, p_a^b||L_{Q,b}|, (A, \preceq))$, which is uniquely determined by Q, is a surjective simplicial sub-inverse system of $\mathbf{X}$. Moreover, for each $x \in Q$, $\mathbf{X}_x$ (see Lemma 2.2) *is a sub-inverse system of $\mathbf{X}_Q$ with $x \in \lim \mathbf{X}_x$, so $Q \subset \lim \mathbf{X}_Q$.*

Proof. Parts (1) and (2) are obviously true. To obtain (3), suppose that $a \preceq b$. First we show that $p_a^b(|L_{Q,b}|) \subset |L_{Q,a}|$. Suppose that $\tau \in L_{Q,b}$, i.e., τ is a face of an element $\sigma_{y,b} \in M_{Q,b}$. Then Lemma 2.2 shows that $p_a^b(\sigma_{y,b}) = \sigma_{y,a} \in M_{Q,a}$. Since $p_a^b(\tau)$ is a face of $\sigma_{y,a}$, then $p_a^b(\tau) \in L_{Q,a}$, so $p_a^b(\tau) \subset |L_{Q,a}|$. Now we show the opposite inclusion, $|L_{Q,a}| \subset p_a^b(|L_{Q,b}|)$. Suppose that $\tau \in L_{Q,a}$. Then there exists $y \in Q$ such that τ is a face of $\sigma_{y,a}$. As before, we know that $p_a^b(\sigma_{y,b}) = \sigma_{y,a}$; hence $\tau \subset \sigma_{y,a} = p_a^b(\sigma_{y,b}) \subset p_a^b(|L_{Q,b}|)$, which proves the desired inclusion. □

Definition 2.6. Let $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ be a simplicial inverse system, $X = \lim \mathbf{X}$, and $Q \subset X$. Then we shall refer to the uniquely determined inverse system $\mathbf{X}_Q$ of Lemma 2.5 as the **trace of** Q in $\mathbf{X}$.

Applying Lemmas 2.5 and 2.2, one arrives at the next result.

Proposition 2.7. *If $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ is a simplicial inverse system and $X = \lim \mathbf{X}$, then $\mathbf{X}_X$, the trace of X in $\mathbf{X}$, is a surjective simplicial sub-inverse system of $\mathbf{X}$ with $\lim \mathbf{X}_X = X$. This shows that X can be represented as the limit of a surjective simplicial inverse system.*

3. Topological Cells in Inverse Limits

In Corollary 3.3 we shall characterize the conditions under which the inverse limit of a simplicial inverse sequence contains a topological n-cell. The same will be done in Corollary 3.5 for a simplicial inverse system in which the coordinate spaces are finite polyhedra. For inverse sequences, we will make use of the class of stratifiable spaces; such spaces are convenient for applications when considering limits of inverse sequences. An exposition of generalized metrizable spaces, including stratifiable spaces, is given by G. Gruenhage [3] in the Handbook of Set-Theoretic Topology. In that work, it is assumed that all spaces under consideration are T_1 and regular. But for our purposes, we will only require that they be T_1.

We note that stratifiable spaces were first called M_3-spaces, but the term stratifiable was introduced in [1] and this nomenclature became standard thenceforward. Lemma 3.1 contains a list of properties of stratifiable spaces. Let us first see which ones can be verified by reference to page numbers from [3]. The definition is given on page 426; we shall not repeat it here. Using that definition and the T_1 property, it is easy to prove that stratifiable spaces are regular; hence they are Hausdorff. Theorem 5.7 on page 457 gives us paracompactness, and Theorem 5.10 on page 458 shows that they are hereditarily stratifiable and countably productive. Hence the limit of an inverse sequence of stratifiable spaces is stratifiable. Corollary 5.12(ii) on page 459 gives us that metrizable spaces are stratifiable. So the only statements in Lemma 3.1 yet to be verified are the one in (4) concerning $|K|_{\mathrm{CW}}$, (5), and (7). We need to get at these from other references.

Every polyhedron $|K|_{\mathrm{CW}}$ has the structure of a CW-complex. If one views the Introduction of [2] (see Corollary 8.6), one can see that all CW-complexes and hence all polyhedra are stratifiable spaces. This gives us the first part of (4). The main result of [5] (see also [6]) shows that covering dimension dim is preserved in the inverse limit of an inverse sequence of stratifiable spaces, so (7) is established. We get (5) from Theorem 3.6 of [4].

Lemma 3.1. *The following are some facts about stratifiable spaces.*

1. *Every stratifiable space is paracompact and Hausdorff.*
2. *Every subspace of a stratifiable space is stratifiable.*
3. *All metrizable spaces are stratifiable.*
4. *For each simplicial complex K, both $|K|_{\mathrm{CW}}$ and $|K|_m$ are stratifiable.*
5. *If $Y \subset X$ and X is a stratifiable space, then $\dim Y \leq \dim X$.*
6. *The limit of an inverse sequence of stratifiable spaces is stratifiable.*
7. *If $\mathbf{X} = (X_i, p_i^{i+1})$ is an inverse sequence of stratifiable spaces, $X = \lim \mathbf{X}$, $n \geq 0$, and for each i, $\dim X_i \leq n$, then $\dim X \leq n$.*

Proposition 3.2. *Let $\mathbf{X} = (|K_i|_{\mathrm{CW}}, p_i^{i+1})$ be a simplicial inverse sequence, $X = \lim \mathbf{X}$, and $n \in \mathbb{N}$. If $\dim X \geq n$, then there exist $i_0 \in \mathbb{N}$ and a sequence $(\tau_i)_{i \geq i_0}$ such that for each $i \geq i_0$, τ_i is an n-simplex of K_i and p_i^{i+1} carries τ_{i+1}*

topologically onto τ_i. The same is true if we replace the topology CW*, where it appears above, by the metric topology m.*

Proof. Applying Proposition 2.7, there is no loss of generality in assuming that p_i^{i+1} is surjective for all i. Using Lemma 3.1(4,6), one sees that X is stratifiable. An application of Lemma 3.1(7) shows this: if it is true that for all i, $\dim |K_i| < n$, then one would have that $\dim X < n$. So there is a first $i_0 \in \mathbb{N}$ with $\dim K_{i_0} \geq n$. Let τ_{i_0} be a simplex of K_{i_0} with $\dim \tau_{i_0} = n$. Using the fact that for each $i \geq i_0$, p_i^{i+1} is simplicial and surjective, one can choose a sequence $(\tau_i)_{i \geq i_0}$ as requested. The same argument can be applied if we replace the topology CW, where it appears, by the metric topology m. □

We obtain a corollary to Lemma 3.1(4,5) and Proposition 3.2.

Corollary 3.3. *Let $\mathbf{X} = (|K_i|_{\mathrm{CW}}, p_i^{i+1})$ be a simplicial inverse sequence, $X = \lim \mathbf{X}$, and $n \in \mathbb{N}$. Then X contains a topological n-cell if and only if $\dim X \geq n$. The same is true if we replace the topology* CW*, where it appears, by the metric topology m.*

Proposition 3.4. *Let $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ be a simplicial inverse system where all the $|K_a|$ are finite polyhedra, $X = \lim \mathbf{X}$, and $n \in \mathbb{N}$. If $\dim X \geq n$, then there exists $d \in A$ such that for each $a \in A$ with $d \preceq a$, there is an n-simplex τ_a of K_a such that if $b \in A$ with $a \preceq b$, then p_a^b carries τ_b topologically onto τ_a. Thus, X contains a topological n-cell.*

Proof. We may assume that $(A, \preceq)$ has no upper bound. Applying Proposition 2.7, there is no loss of generality in assuming that p_a^b is surjective for all $a \preceq b$. It is moreover true that X is a compact Hausdorff space. Since $\dim X \geq n$, there has to be a cofinal subset A_0 of A such that $\dim K_a \geq n$ for all $a \in A_0$. We may as well require that A has this property from the outset. Fix $d \in A$. Then the set of $a \in A$ with $d \preceq a$ is cofinal in A, so we shall assume that for all $a \in A$, $d \preceq a$.

Now fix an n-simplex τ_d in K_d, let $x_{\tau_d} \in \operatorname{int} \tau_d$, and $H_d = \{x_{\tau_d}\}$. For each $a \in A$, there is at least one n-simplex $\tau \in K_a$ such that $p_d^a(\tau) = \tau_d$. Let $\mathcal{F}_a$ be the collection of such n-simplexes, and for each $\tau \in \mathcal{F}_a$, select the unique element $x_\tau \in \operatorname{int} \tau$ with $p_d^a(x_\tau) = x_{\tau_d}$. Denote $H_a = \{x_\tau \,|\, \tau \in \mathcal{F}_a\}$. Then for all $a \in A$, H_a is a finite, nonempty subset of $|K_a|$, and if $u \in H_a$, then $p_d^a(u) = x_{\tau_d}$.

We claim that if $a \preceq b$, then $p_a^b(H_b) \subset H_a$. For let $\tau \in \mathcal{F}_b$; we must show that $p_a^b(x_\tau) \in H_a$. Now $p_d^a \circ p_a^b(x_\tau) = p_d^b(x_\tau) = x_{\tau_d}$. Also, $p_d^b(\tau) = \tau_d$. It follows that $\tau^* = p_a^b(\tau)$ is an n-simplex of K_a and $p_d^a(\tau^*) = \tau_d$. Thus, $\tau^* \in \mathcal{F}_a$ and $p_a^b(x_\tau) = x_{\tau^*} \in H_a$ as required. From this we get a sub-inverse system $\mathbf{H} = (H_a, p_a^b|H_b, (A \preceq))$ of $\mathbf{X}$ consisting of nonempty discrete finite sets H_a. Thus $\lim \mathbf{H} \neq \varnothing$. Select $y \in \lim \mathbf{H} \subset \lim \mathbf{X}$. From Lemma 2.2, the trace of y in $\mathbf{X}$, $\mathbf{X}_y = (\sigma_{y,a}, p_a^b|\sigma_{y,b}, (A, \preceq))$ is a surjective simplicial sub-inverse system of $\mathbf{X}$. Since $\dim \sigma_{y,a} = n$ for all a, then each $p_a^b|\sigma_{y,b} : \sigma_{y,b} \to \sigma_{y,a}$ is a homeomorphism. Clearly, $\lim \mathbf{X}_y \subset \lim \mathbf{X}$ is a topological n-cell. □

Corollary 3.5. *Let* $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ *be a simplicial inverse system where all the* $|K_a|$ *are finite polyhedra,* $X = \lim \mathbf{X}$*, and* $n \in \mathbb{N}$*. Then* X *contains a topological* n*-cell if and only if* $\dim X \geq n$*.*

4. Hilbert Cubes in Limits of Inverse Sequences

The main result of this section is Theorem 4.13. It characterizes when the limit of a simplicial inverse sequence must contain a copy of the Hilbert cube. First, let us review some concepts from dimension theory. Recall that an infinite-dimensional space is called *countable-dimensional* if it can be written as the union of subspaces X_n, $n \in \mathbb{N}$, each X_n having dimension $\leq n$. It is called *strongly countable-dimensional* if it can be written as the union of closed subspaces X_n, $n \in \mathbb{N}$, each X_n having dimension $\leq n$. Of course, strongly countable-dimensional spaces are countable-dimensional.

From Corollaries 3.3 and 3.5, respectively, we get Propositions 4.1 and 4.2.

Proposition 4.1. *Let* $\mathbf{X} = (|K_i|_{\mathrm{CW}}, p_i^{i+1})$ *be a simplicial inverse sequence and* $X = \lim \mathbf{X}$*. If* $\dim X = \infty$*, then* X *contains a strongly countable dimensional subspace* $Y = \bigcup\{Y_i \,|\, i \in \mathbb{N}\}$ *such that for each* i*,* Y_i *is a topological* i*-cell. The same is true if we replace the topology* CW*, where it appears above, by the metric topology* m*.*

Proposition 4.2. *Let* $\mathbf{X} = (|K_a|_{\mathrm{CW}}, p_a^b, (A, \preceq))$ *be a simplicial inverse system and* $X = \lim \mathbf{X}$*. If all the* $|K_a|$ *are finite polyhedra and* $\dim X = \infty$*, then* X *contains a strongly countable dimensional subspace* $Y = \bigcup\{Y_i \,|\, i \in \mathbb{N}\}$ *such that for each* i*,* Y_i *is a topological* i*-cell.*

As usual, $I = [0, 1]$, the unit interval. We shall denote the Hilbert cube as I^∞, that is, $I^\infty = \prod\{I_i \,|\, i \in \mathbb{N}\}$ where for each i, $I_i = I$. For each $i \in \mathbb{N}$, let $p_i^{i+1} : I^{i+1} \to I^i$ be the i-coordinate projection. Remember that strongly infinite-dimensional spaces are not countable-dimensional. Since I^∞ is strongly infinite-dimensional, it is not countable-dimensional. One may consult [10] for more information on this subject.

Lemma 4.3. *Let* $\mathbf{G} = (I^i, p_i^{i+1})$ *be the inverse sequence having the property that for each* i*,* $p_i^{i+1} : I^{i+1} \to I^i$ *is the coordinate projection. Then* $\lim \mathbf{G} \cong I^\infty$*.*

Proof. Since both I^∞ and $\lim \mathbf{G}$ are compact metrizable spaces, it is sufficient to find a bijective map from I^∞ to $\lim \mathbf{G}$. Define a map $h : I^\infty \to \lim \mathbf{G}$ by setting $h(x_1, x_2, x_3, \dots) = (x_1, (x_1, x_2), (x_1, x_2, x_3), \dots)$. Surely h is a map; we leave it to the reader to show that h is a bijection. □

Whenever $\mathcal{V}$ is the vertex set of a simplex σ, then an arbitrary element x of σ will be written $x = \sum\{x_v v \,|\, v \in \mathcal{V}\}$, where for each $v \in \mathcal{V}$, x_v is the v-barycentric coordinate of x.

Lemma 4.4. *Let* $n \in \mathbb{N}$*,* σ *be an* n*-simplex with vertex set* $\mathcal{V}$*,* τ_0 *an* $(n-1)$*-face of* σ*,* $\mathcal{W}$ *the vertex set of* τ_0*,* $v \in \mathcal{V} \setminus \mathcal{W}$*, and* $\mu : \sigma \to \tau_0$ *a simplicial retraction. Then* $\mu(u) = u$ *for each* $u \in \mathcal{W}$*, and there is a unique* $w \in \mathcal{W}$ *with* $\mu(v) = w$*.*

Indeed, if $x = \sum\{x_v v \,|\, v \in \mathcal{V}\} \in \sigma$*, then* $\mu(x) = \sum\{b_u u \,|\, u \in \mathcal{W}\} \in \tau_0$ *where* $b_u = x_v + x_w$ *if* $u = w$*, and* $b_u = x_u$ *otherwise.*

Lemma 4.5. *Let* $n \in \mathbb{N}$*,* σ *be an* n*-simplex,* τ *an* $(n-1)$*-simplex, and* $q : \sigma \to \tau$ *a simplicial surjection. Then there exist a unique* $(n-1)$*-face* τ_0 *of* σ*, a unique simplicial retraction* $\mu : \sigma \to \tau_0$*, and a unique simplicial isomorphism* $q_0 : \tau_0 \to \tau$*, such that* $q = q_0 \circ \mu$*.*

Lemma 4.6. *Let* $n \in \mathbb{N}$*,* σ *be an* n*-simplex,* τ_0 *an* $(n-1)$*-face of* σ*, and* $\mu : \sigma \to \tau_0$ *a simplicial retraction of* σ *to* τ_0*. Suppose that* $D \subset \operatorname{int} \tau_0$ *is nonempty and compact. Let* $\mathcal{V}$*,* $\mathcal{W}$*,* v*, and* w *come from* Lemma 4.4*. We claim that for any neighborhood* U *of* $\partial\sigma$ *in* σ*, there is an embedding* $H : D \times I \to U \cap \operatorname{int} \sigma$ *such that* $(\mu|\operatorname{im}(H)) \circ H = p : D \times I \to D$*, where* $p : D \times I \to D$ *is the coordinate projection.*

Proof. Let $(x,t) \in D \times I$, $x = \sum\{x_u u \,|\, u \in \mathcal{W}\} \in D \subset \operatorname{int} \tau_0$. Define $H(x,t) \in \sigma$ so that its v-barycentric coordinate is $(1-t)x_w$, its w-barycentric coordinate is tx_w, and for any $u \in \mathcal{V} \setminus \{v,w\}$, its u-barycentric coordinate is x_u. Then clearly $H : D \times I \to \sigma$ is a map. To show that H is injective, let $y = \sum\{y_u u \,|\, u \in \mathcal{W}\} \in D$, $\{t,s\} \subset I$, and $(x,t) \neq (y,s)$. If $u \in \mathcal{W} \setminus \{w\}$, and $x_u \neq y_u$, then $H(x,t) \neq H(y,s)$ independently of t and s. Hence we may as well assume that $x_u = y_u$ for all $u \in \mathcal{W} \setminus \{w\}$. Suppose that $H(x,t) = H(y,s)$. If $t = s$, then $x \neq y$, that is, $x_w \neq y_w$. By the definition of H, $(1-t)x_w = (1-t)y_w$ and $tx_w = ty_w$. Since one of $\{1-t, t\}$ does not equal 0, then $x_w = y_w$, a contradiction. Hence $t \neq s$. Since $D \subset \operatorname{int} \tau_0$, then $x_w \neq 0$, so $tx_w \neq sx_w$. This implies that $t = s$, another contradiction. Therefore $H(x,t) \neq H(y,s)$. We have demonstrated that H is injective which shows that H is an embedding because of compactness. One easily checks that $(\mu|\operatorname{im}(H)) \circ H = p : D \times I \to D$.

Notice that for $x \in D \subset \operatorname{int} \tau_0$ as above, for all $u \in \mathcal{W}$, $x_u > 0$. This is true in particular if $u = w$. If $t \notin \{0,1\}$, both $(1-t)x_w > 0$ and $tx_w > 0$. Hence for all $u \in \mathcal{W}$, the u-barycentric coordinates of $H(x,t)$ are > 0. Therefore if $0 < a < b < 1$ and we restrict H to $D \times [a,b]$, we get an embedding of $D \times [a,b]$ into $\operatorname{int} \sigma$. But, the v-barycentric coordinate of $H(x,1)$ equals 0. So $H(D \times \{1\}) \subset \partial\sigma$. Taking a sufficiently close to 1, we get that $H(D \times [a,b]) \subset U \cap \operatorname{int} \sigma$. It is now simply a matter of reparameterizing $[a,b]$ so that it is replaced by $[0,1]$, and we have our proof. □

Lemma 4.7. *Let* $n \in \mathbb{N}$*,* σ *be an* n*-simplex,* τ *an* $(n-1)$*-simplex, and* $q : \sigma \to \tau$ *a simplicial surjection. Suppose that* E *is a nonempty compact subset of* $\operatorname{int} \tau$*. Then for any neighborhood* U *of* $\partial\sigma$ *in* σ*, there is an embedding* $H^* : E \times I \to U \cap \operatorname{int} \sigma$ *such that* $(q|\operatorname{im}(H^*)) \circ H^* = p : E \times I \to E$*, where* $p : E \times I \to E$ *is the coordinate projection.*

Proof. Apply Lemma 4.5 to $q : \sigma \to \tau$. Let τ_0 be the unique $(n-1)$-face of σ, $\mu : \sigma \to \tau_0$ the unique simplicial retraction, and $q_0 : \tau_0 \to \tau$ the unique simplicial isomorphism such that $q = q_0 \circ \mu$. Put $D = q_0^{-1}(E) \subset \operatorname{int} \tau_0$. Apply Lemma 4.6 to get an embedding $H : D \times I \to U \cap \operatorname{int} \sigma$ having the property that $(\mu|\operatorname{im}(H)) \circ H = p : D \times I \to D$, where $p : D \times I \to D$ is the coordinate projection. Define $H^* : E \times I \to \operatorname{int}\sigma$ by $H^*(e,t) = H(q_0^{-1}(e),t)$. Surely H^* is an embedding of $E \times I$ into $U \cap \operatorname{int}\sigma$. Suppose that $(e,t) \in E \times I$. Then $q \circ H^*(e,t) = q_0 \circ \mu \circ H^*(e,t) = q_0 \circ \mu \circ H(q_0^{-1}(e),t) = q_0 \circ p(q_0^{-1}(e),t) = q_0 \circ q_0^{-1}(e) = e$. □

Lemma 4.8. *Let $m < n \in \mathbb{N}$ and $\{\sigma_i \,|\, 0 \le i \le n-m\}$ be a set such that for each $0 \le i \le n-m$, σ_i is an $(m+i)$-simplex. For each $1 \le i \le n-m$, let $q_i : \sigma_i \to \sigma_{i-1}$ be a simplicial surjection and put $q = q_1 \circ \cdots \circ q_{n-m} : \sigma_{n-m} \to \sigma_0$. Let E be a nonempty compact subset of $\operatorname{int}\sigma_0$ and U a neighborhood of $\partial\sigma_{n-m}$ in σ_{n-m}. Then there is an embedding $H^* : E \times I^{n-m} \to U \cap \operatorname{int}\sigma_{n-m}$ such that $(q|\operatorname{im}(H^*)) \circ H^* = p : E \times I^{n-m} \to E$, where $p : E \times I^{n-m} \to E$ is the coordinate projection.*

Proof. An application of Lemma 4.7 shows that this result is true in every case where $n - m = 1$. Suppose that $k \in \mathbb{N}$, and the lemma is true in every case where $n - m = k$. Now assume that $n - m = k + 1$ and we are given the above data, only this time with one more map in the composition. Note that in this setting, $q = q' \circ q_{k+1}$ where $q_{k+1} : \sigma_{k+1} \to \sigma_k$, $\dim\sigma_{k+1} = \dim\sigma_k + 1$, $q' = q_1 \circ \cdots \circ q_k : \sigma_k \to \sigma_0$, and $k = n-(m+1) > 0$. Also, U is a neighborhood of $\partial\sigma_{k+1}$ in σ_{k+1}. Thus, $m+1 < n$, so we may apply the inductive hypothesis to the map q'. This gives us an embedding $H : E \times I^k \to \operatorname{int}\sigma_k$ such that $(q'|\operatorname{im}(H)) \circ H = p' : E \times I^k \to E$, where $p' : E \times I^k \to E$ is the coordinate projection.

We now have the nonempty compact subset $\operatorname{im} H \subset \operatorname{int}\sigma_k$ and of course $k + 1 - k = 1$. So we may apply the fact that our result is true for $n = k+1$, $m = k$. This gives us an embedding $H' : (\operatorname{im} H) \times I$ into $U \cap \operatorname{int}\sigma_{k+1}$ such that $(q_{k+1}|\operatorname{im}(H')) \circ H' = p^* : (\operatorname{im} H) \times I \to \operatorname{im} H$, where $p^* : (\operatorname{im} H) \times I \to \operatorname{im} H$ is the coordinate projection. Define $H^* : E \times I^k \times I \to U \cap \operatorname{int}\sigma_{k+1}$ by $H^*(e,s,t) = H'(H(e,s),t)$. It follows that H^* is an embedding.

We must prove that $(q|\operatorname{im}(H^*)) \circ H^* = p : E \times I^{k+1} \to E$, where $p : E \times I^{k+1} \to E$ is the coordinate projection. Let $(e,s,t) \in E \times I^k \times I$. Then $q \circ H^*(e,s,t) = q \circ H'(H(e,s),t) = q' \circ q_{k+1} \circ H'(H(e,s),t) = q' \circ p^*(H(e,s),t) = q' \circ H(e,s) = p'(e,s) = e$. Our proof is complete. □

Applying Lemmas 4.5 and 4.8, one obtains a corollary.

Corollary 4.9. *Let σ and τ be simplexes such that $\dim\tau = m < \dim\sigma = n$, suppose that $p : \sigma \to \tau$ is a simplicial surjection, E is a compact subset of $\operatorname{int}\tau$ and U is a neighborhood of $\operatorname{bd}\sigma$ in σ. Then there exists an embedding $H : E \times I^{n-m} \to U \cap \operatorname{int}\sigma$ such that $p \circ H : E \times I^{n-m} \to \tau$ is the coordinate projection $E \times I^{n-m}$ to E.*

Proposition 4.10. *Suppose that $\mathbf{S} = (\sigma_i, q_i^{i+1})$ is a surjective simplicial inverse sequence such that for each i, σ_i is an i-simplex. Then $\lim\mathbf{S}$ contains an embedded copy of I^∞.*

Proof. Let $E \subset \operatorname{int}\sigma_1$ be a closed interval, and identify E with I. Apply Lemma 4.7 in such a way that $I \times I \subset \operatorname{int}\sigma_2$ and $q_1^2|I \times I : I \times I \to I \subset \operatorname{int}\sigma_1$

is the coordinate projection $(t_1, t_2) \mapsto t_1$. Next apply Lemma 4.7 again in such a way that $I^2 \times I \subset \operatorname{int} \sigma_3$ and $q_2^3 | I^2 \times I : I^2 \times I \to I^2 \subset \operatorname{int} \sigma_2$ is the coordinate projection $(t_1, t_2, t_3) \mapsto (t_1, t_2)$.

Continuing recursively in this manner, we land up with a sub-inverse sequence of $\mathbf{S}$ of the form $\mathbf{G} = (I^i, p_i^{i+1})$ from Lemma 4.3. Therefore $\lim \mathbf{G} \cong I^\infty \subset \lim \mathbf{S}$ as requested. □

Corollary 4.11. *Suppose that* $\mathbf{S} = (\sigma_i, q_i^{i+1})$ *is a surjective simplicial inverse sequence such that for each* i, σ_i *is a simplex, and there exists an increasing sequence* (n_i) *in* $\mathbb{N}$ *such that for each* i, $\dim \sigma_{n_i} < \dim \sigma_{n_{i+1}}$. *Then* $\lim \mathbf{S}$ *contains an embedded copy of* I^∞.

Proof. Since the sequence (n_i) is increasing, we may replace $\mathbf{S}$ with the inverse sequence $(\sigma_{n_i}, q_{n_i}^{n_{i+1}})$ whose inverse limit is homeomorphic to $\lim \mathbf{S}$. To conserve notation, let us assume that the given inverse sequence $\mathbf{S} = (\sigma_i, q_i^{i+1})$ already has the property that $\dim \sigma_i < \dim \sigma_{i+1}$ for all i. One may also assume that $1 \leq \dim \sigma_1$. Select a 1-face τ_1 of σ_1. Choose a 2-face τ_2 of σ_2 with $q_1^2(\tau_2) = \tau_1$. Similarly, choose a 3-face τ_3 of σ_3 with $q_2^3(\tau_3) = \tau_2$. This process can be continued recursively so that we end up with a sequence (τ_i) having the property that for each i, $\dim \tau_i = i$, τ_i is a face of σ_i, and $q_i^{i+1} | \tau_{i+1} : \tau_{i+1} \to \tau_i$ is a simplicial surjection. The surjective simplicial sub-inverse sequence $\mathbf{S}_0 = (\tau_i, q_i^{i+1} | \tau_{i+1})$ of $\mathbf{S}$ replicates the inverse sequence in Proposition 4.10, so I^∞ embeds in $\lim \mathbf{S}_0$ which in turn embeds in $\lim \mathbf{S}$. □

Lemma 4.12. *Let* $\mathbf{X} = (|K_i|_{\mathrm{CW}}, p_i^{i+1})$ *be a simplicial inverse sequence, and put* $X = \lim \mathbf{X}$. *Suppose that* X *contains a strongly infinite-dimensional subspace* Q. *Then there exist* $x \in Q$ *and an increasing sequence* (n_i) *in* $\mathbb{N}$, *so that the trace* $\mathbf{X}_x$ *of* x *in* $\mathbf{X}$ *has the property that for each* i, $\dim \sigma_{x,n_i} < \dim \sigma_{x,n_{i+1}}$. *The same is true if we replace the topology* CW, *where it appears above, by the metric topology* m.

Proof. For each $x \in Q \subset X$, let $\mathbf{X}_x$ be the trace of x in $\mathbf{X}$. Then for all i, $\sigma_{x,i} \in K_i$ and $p_i^{i+1}(\sigma_{x,i+1}) = \sigma_{x,i}$, so $\dim \sigma_{x,i} \leq \dim \sigma_{x,i+1}$; moreover, $x \in \lim \mathbf{X}_x$. Let us suppose, for the purpose of reaching a contradiction, that for all $x \in Q$, there exists $n_x \in \mathbb{N}$ such that $\dim \sigma_{x,i} \leq n_x$ for all i. For each $n \in \mathbb{N}$, let $Q_n = \{x \in Q \mid n_x \leq n\}$. Then $Q = \bigcup \{Q_n \mid n \in \mathbb{N}\}$.

Fix $n \in \mathbb{N}$, and for each $i \in \mathbb{N}$, let $M_{Q_n,i}$ be as in Definition 2.4. Then all the simplexes in $M_{Q_n,i}$ have dimension $\leq n$. So by Lemma 2.5(2), $\dim L_{Q_n,i} \leq n$. Applying Proposition 2.7, we get the sub-inverse sequence $\mathbf{X}_{Q_n} = (|L_{Q_n,i}|, p_i^{i+1} || L_{Q_n,i+1}|)$ of $\mathbf{X}$, with $Q_n \subset X_n = \lim \mathbf{X}_{Q_n}$. Surely X_n is a stratifiable space and $\dim X_n \leq n$. Thus, $\dim(Q_n \cap X_n) \leq n$. Hence $Q = \bigcup \{Q_n \cap X_n \mid n \in \mathbb{N}\}$ is countable-dimensional, which is false. This same argument works if we replace the topology CW, where it appears, by the metric topology m. Our proof is complete. □

Putting together Corollary 4.11 and Lemma 4.12, we obtain a theorem.

Theorem 4.13. *Let* $\mathbf{X} = (|K_i|_{\mathrm{CW}}, p_i^{i+1})$ *be a simplicial inverse sequence, and put* $X = \lim \mathbf{X}$. *Then* X *contains an embedded copy of* I^∞ *if and only if there is a collection* $\{\sigma_i \,|\, i \in \mathbb{N}\}$ *and an increasing sequence* (n_i) *in* $\mathbb{N}$, *such that for each* i,

(1) σ_i *is a simplex of* K_i,
(2) $p_i^{i+1}(\sigma_{i+1}) = \sigma_i$, *and*
(3) $\dim \sigma_{n_i} < \dim \sigma_{n_{i+1}}$.

The same is true if we replace the topology CW, *where it appears above, by the metric topology* m.

5. Strongly Infinite Dimensional Sets in Limits of Inverse Systems of Finite Polyhedra

We present a result for inverse systems of finite polyhedra that is parallel to Lemma 4.12. We however *do not have* a result that is similar to that of Theorem 4.13.

Proposition 5.1. *Let* $\mathbf{X} = (|K_a|, p_a^b, (A, \preceq))$ *be a simplicial inverse system where all the* $|K_a|$ *are finite polyhedra, and let* $X = \lim \mathbf{X}$. *Suppose that* X *contains a strongly infinite-dimensional closed subspace* Q. *Then there exists* $x \in X$ *(indeed,* $x \in Q$*) so that the trace* $\mathbf{X}_x$ *of* x *in* $\mathbf{X}$ *satisfies the property that for each* $a \in A$ *and* $n \in \mathbb{N}$, *there exists* $a \preceq b$ *such that* $\dim \sigma_{x,b} \geq n$. *Hence there exists a sequence* (a_i) *in* A *such that for each* i, $a_i \preceq a_{i+1}$, $a_i \neq a_{i+1}$, *and* $\dim \sigma_{a_i} < \dim \sigma_{a_{i+1}}$.

Proof. Since X contains a strongly infinite-dimensional closed subspace, then $(A, \preceq)$ has no upper bound. For each $x \in Q \subset X$, let $\mathbf{X}_x$ be the trace of x in $\mathbf{X}$. Let us suppose, for the purpose of reaching a contradiction, that for all $x \in Q$, there exist $a_x \in A$ and $n_x \in \mathbb{N}$ such that for all $a_x \preceq b$, $\dim \sigma_{x,b} \leq n_x$. For each $n \in \mathbb{N}$, let $Q_n = \{x \in Q \,|\, n_x \leq n\}$. Then $Q = \bigcup\{Q_n \,|\, n \in \mathbb{N}\}$.

Fix $n \in \mathbb{N}$, and for each $a \in A$, let $M_{Q_n,a}$ be as in Definition 2.4. Then whenever $a_x \preceq b$, by Lemma 2.5(2), $\dim L_{Q_n,b} \leq n$. One should note that $\{b \in A \,|\, a_x \preceq b\}$ is cofinal in A. Applying Definition 2.6, we get the sub-inverse system $\mathbf{X}_{Q_n} = (|L_{Q_{n,a}}|, p_a^b\big||L_{Q_{n,b}}|, (A \preceq))$ of $\mathbf{X}$, with $Q_n \subset X_n = \lim \mathbf{X}_{Q_n}$. Surely, X_n is a compact Hausdorff space and $\dim X_n \leq n$. Hence $Q = \bigcup\{Q_n \cap X_n \,|\, n \in \mathbb{N}\}$ is strongly countable-dimensional, which is false since Q is strongly infinite-dimensional. Our proof is complete. □

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