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Characterizing meager paratopological groups

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Abstract

We prove that a Hausdorff paratopological group G is meager if and only if there are a nowhere dense subset $A \subset G$ and a countable set $C \subset G$ such that $CA = G = AC$.

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1. Introduction

Trying to find a counterpart of the Lindeföf property in the category of topological groups I.Guran [7] introduced the notion of an ω-bounded group which turned out to be very fruitful in topological algebra, see [9]. We recall that a topological group G is *ω-bounded* if for each non-empty open subset $U \subset G$ there is a countable subset $C \subset G$ such that $CU = G = UC$.

A similar approach to the Baire category leads us to the notion of an *shift-meager* (*shift-Baire*) group. This is a topological group that can(not) be written as the union of countably many translation copies of some fixed nowhere dense subset.

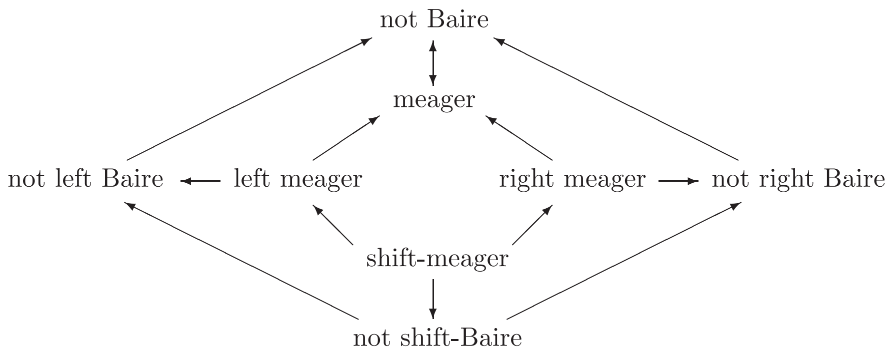
The notion of a shift-meager (shift-Baire) group can be defined in a more general context of semitopological groups, that is, groups G endowed with a shift-invariant topology τ. The latter is equivalent to saying that the group operation $\cdot : G \times G \to G$ is separately continuous. If this operation is jointly continuous, then (G, τ) is called a *paratopological group*, see [1].

A semitopological group G is defined to be

- *left meager* (resp. *right meager*) if $G = CA$ (resp. $G = AC$) for some nowhere dense subset $A \subset X$ and some countable subset $C \subset G$;
- *shift-meager* if G is both left and right meager;

- *left Baire* (resp. *right Baire*) if for every open dense subset $U \subset X$ and every countable subset $C \subset G$ the intersection $\bigcap_{x\in C} xU$ (resp. $\bigcap_{x\in C} Ux$) is dense in G;
- *shift-Baire* if G is both left and right Baire.

For semitopological groups those notions relate as follows:



The following theorem implies that for Hausdorff paratopological groups all the eight properties from this diagram are equivalent.

Theorem 1.1. *A Hausdorff paratopological group G is meager if and only if G is shift-meager.*

This theorem will be proved in Section 4. The proof is based on Theorem 2.1 giving conditions under which a meager semitopological group is left (right) meager and Theorem 3.2 describing some oscillator properties of 2-saturated Hausdorff paratopological groups.

2. Shift-meager semitopological groups

In this section we search for conditions under which a given meager semitopological group is left (right) meager.

Following [3], [4] and [5], [6], we define a subset $A \subset G$ of a group G to be

- *left large* (resp. *right large*) if $G = FA$ (resp. $G = AF$) for some finite subset $F \subset G$;
- *left P-small* (resp. *right P-small*) if there is an infinite subset $B \subset G$ such that the indexed family $\{bA\}_{b\in B}$ (resp. $\{Ab\}_{b\in B}$) is disjoint.

Theorem 2.1. *A meager semitopological group G is left (right) meager provided one of the following conditions holds:*

1. *G contains a non-empty open left (right) P-small subset;*
2. *G contains a sequence $(U_n)_{n\in\omega}$ of pairwise disjoint open left (right) large subsets;*
3. *G contains sequences of non-empty open sets $(U_n)_{n\in\omega}$ and points $(g_n)_{n\in\omega}$ such that the sets $g_nU_nU_n^{-1}$ (resp. $U_n^{-1}U_ng_n$), $n \in \mathbb{N}$, are pairwise disjoint.*

Proof. (1_l) Assume that $U \subset G$ is a non-empty open left P-small subset. We may assume that U is a neighborhood of the neutral element e of G. It follows that there is a countable subset $B = \{b_n\}_{n\in\omega} \subset G$ such that $b_nU \cap b_mU = \varnothing$ for any distinct numbers $n \neq m$. The countable set B generates a countable subgroup H of G. By an H-*cylinder* we shall understand an open subset of the form HVg where $g \in G$ and $V \subset U$ is a neighborhood of e. Let $\mathcal{U} = \{HV_\alpha g_\alpha : \alpha \in A\}$ be a maximal disjoint family of H-cylinders in G (such a family exists by the Zorn Lemma).

We claim that $\cup\mathcal{U}$ is dense in G. Assuming the converse, we could find a point $g \in G \setminus \cup\mathcal{U}$ and a neighborhood $V \subset U$ of e such that $Vg \cap \cup\mathcal{U}$. Taking into account that $H \cdot (\cup\mathcal{U}) = \cup\mathcal{U}$, we conclude that $HVg \cap \cup\mathcal{U} = \varnothing$ and hence $\mathcal{U} \cup \{HVg\}$ is a disjoint family of H-cylinders that enlarges the family $\mathcal{U}$, which contradicts the maximality of $\mathcal{U}$. Therefore $\cup\mathcal{U}$ is dense in G and hence $G \setminus \cup\mathcal{U}$ is a closed nowhere dense subset of G.

The space G, being meager, can be written as the union $G = \bigcup_{n\in\omega} M_n$ of a sequence $(M_n)_{n\in\omega}$ of nowhere dense subsets of G. It is easy to see that the set

$$M = (G \setminus \cup\mathcal{U}) \cup \bigcup_{\alpha\in A} \bigcup_{n\in\omega} b_n(M_n \cap V_\alpha g_\alpha)$$

is nowhere dense in G and $G = HM$, witnessing that G is left meager.

(2_l) Assume that G contains a sequence $(U_n)_{n\in\omega}$ of pairwise disjoint open left large subsets. For every $n \in \omega$ find a finite subset $F_n \subset G$ with $G = F_n \cdot U_n$. Write $G = \bigcup_{n\in\omega} M_n$ as countable union of nowhere dense subsets and observe that for every $n \in \omega$ the subset $\bigcup_{x\in F_n} x^{-1}(M_n \cap xU_n)$ of U_n is nowhere dense. Since the family $\{U_n\}_{n\in\omega}$ is disjoint, the set

$$M = \bigcup_{n\in\omega} \bigcup_{x\in F_n} x^{-1}(M_n \cap xU_n)$$

is nowhere dense in G. Since $(\bigcup_{n\in\omega} F_n) \cdot M = G$, the semitopological group G is left meager.

(3_l) Assume that $(U_n)_{n\in\omega}$ is a sequence of non-empty open subsets of G and $(g_n)_{n\in\omega}$ is a sequence of points of G such that the sets $g_nU_nU_n^{-1}$, $n \in \omega$, are pairwise disjoint. Using the Zorn Lemma, for every $n \in \omega$ we can choose a maximal subset $F_n \subset G$ such that the indexed family $\{xU_n\}_{x\in F_n}$ is disjoint. If for some $n \in \omega$ the set F_n is infinite, then the set U_n is left P-small and consequently, the group G is left meager by the first item. So, assume that each set F_n, $n \in \omega$, is finite. The maximality of F_n implies that for every $x \in G$ there is $y \in F_n$ such that $xU_n \cap yU_n \neq \varnothing$. Then $x \in yU_nU_n^{-1}$ and hence $G = F_nU_nU_n^{-1}$, which means that the open set $U_nU_n^{-1}$ is left large. Since the family $\{g_nU_nU_n^{-1}\}_{n\in\omega}$ is disjoint, it is legal to apply the second item to conclude that the group G is left meager.

$(1_r)-(3_r)$. The right versions of the items (1)–(3) can be proved by analogy. □

3. Oscillation properties of paratopological groups

In this section we establish some oscillation properties of 2-saturated paratopological groups. First, we recall the definition of oscillator topologies on a given paratopological group (G,τ), see [2] for more details.

Given a subset $U\subset G$, by induction define subsets $(\pm U)^n$ and $(\mp U)^n$, $n\in\omega$, of G letting

$$(\pm U)^0=(\mp U)^0=\{e\}\quad\text{and}\quad (\pm U)^{n+1}=U(\mp U)^n,\ \ (\mp U)^{n+1}=U^{-1}(\pm U)^n$$

for $n\ge 0$. Thus

$$(\pm U)^n=\underbrace{UU^{-1}U\cdots U^{(-1)^{n-1}}}_{n}\quad\text{and}\quad(\mp U)^n=\underbrace{U^{-1}UU^{-1}\cdots U^{(-1)^n}}_{n}.$$

Note that $((\pm U)^n)^{-1}=(\pm U)^n$ if n is even and $((\pm U)^n)^{-1}=(\mp U)^n$ if n is odd.

By an *n-oscillator* (resp. *a mirror n-oscillator*) on a topological group (G,τ) we understand a set of the form $(\pm U)^n$ (resp. $(\mp U)^n$) for some neighborhood U of the unit of G. Observe that each n-oscillator in a paratopological group (G,τ) is a mirror n-oscillator in the mirror paratopological group (G,τ^{-1}) and vice versa: each mirror n-oscillator in (G,τ) is an n-oscillator in (G,τ^{-1}).

By the *n-oscillator topology* on a paratopological group (G,τ) we understand the topology τ_n consisting of sets $U\subset G$ such that for each $x\in U$ there is an n-oscillator $(\pm V)^n$ with $x\cdot(\pm V)^n\subset U$.

Let us recall [8], [1, p. 342] that a paratopological group (G,τ) is *saturated* if each non-empty open set $U\subset G$ has non-empty interior in the mirror topology $\tau^{-1}=\{U^{-1}:U\in\tau\}$. This notion can be generalized as follows.

Define a paratopological group (G,τ) to be *n-saturated* if each non-empty open set $U\in\tau_n$ has non-empty interior in the topology $(\tau^{-1})_n$.

Proposition 3.1. *A paratopological group (G,τ) is 2-saturated if no non-empty open subset $U\subset G$ is P-small.*

Proof. To prove that G is 2-saturated, take any non-empty open set $U_2\in\tau_2$ and find a point $x\in U_2$ and a neighborhood $U\in\tau$ of e such that $xU^2U^{-2}\subset U_2$. By the Zorn Lemma, there is a maximal subset $B\subset G$ such that $bU\cap b'U=\varnothing$ for all distinct points $b,b'\in B$. By our hypothesis, U is not P-small, which implies that the set B is finite. The maximality of B implies that for each $x\in G$ the shift xU meets some shift bU, $b\in B$. Consequently, $x\in bUU^{-1}$ and $G=\bigcup_{b\in B}bUU^{-1}$. It follows that the closure $\overline{UU}^{-1}$ of UU^{-1} in the topology $(\tau^{-1})_2$ has non-empty interior. We claim that $\overline{UU}^{-1}\subset U^2U^{-2}$. Indeed, given any point $z\in\overline{UU}^{-1}$, we conclude that the neighborhood $U^{-1}zU$ of z in the topology $(\tau^{-1})_2$ meets UU^{-1} and hence $z\in U^2U^{-2}$. Now we see that the set

$$U_2\supset xU^2U^{-2}\supset x\overline{UU^{-1}}$$

has non-empty interior in the topology $(\tau^{-1})_2$, witnessing that the group (G,τ) is 2-saturated. $\square$

By Proposition 2 of [2], for each saturated paratopological group (G,τ) the semitopological group (G,τ_2) is a topological group. This results generalizes to n-saturated groups.

Theorem 3.2. *If (G,τ) is an n-saturated paratopological group for some $n\in\mathbb{N}$, then (G,τ_{2n}) is a topological group.*

Proof. According to Theorem 1 of [2], (G,τ_{2n}) is a topological group if and only if for every neighborhood $U\in\tau$ of the neutral element $e\in G$ there is a neighborhood $V\in\tau$ of e such that $(\mp V)^{2n}\subset(\pm U)^{2n}$.

Since the paratopological group (G,τ) is n-saturated, the set $(\pm U)^n\in\tau_2$ contains an interior point x in the mirror topology $(\tau^{-1})_n$. Consequently, there is a neighborhood $V\in\tau$ of e such that $(\mp V)^n x\subset(\pm U)^n$.

Now we consider separately the cases of odd and even n.

1. If n is odd, then applying the operation of the inversion to $(\mp V)^n x\subset(\pm U)^n$, we get $x^{-1}(\pm V)^n\subset(\mp U)^n$ and then

$$(\mp V)^{2n}=(\mp V)^n(\pm V)^n=(\mp V)^n xx^{-1}(\pm V)^n\subset(\pm U)^n(\mp U)^n=(\pm U)^{2n}.$$

2. If n is even, then $(\mp V)^n x\subset(\pm U)^n$ implies $x^{-1}(\mp V)^n\subset(\pm U)^n$ and

$$(\mp V)^{2n}=(\mp V)^n(\mp V)^n=(\mp V)^n xx^{-1}(\mp V)^n\subset(\pm U)^n(\pm U)^n=(\pm U)^{2n}.$$

□

According to [2], for each 1-saturated Hausdorff paratopological groups (G,τ) the group (G,τ_2) is a Hausdorff topological group. For 2-saturated group we have a bit weaker result.

Theorem 3.3. *For any non-discrete Hausdorff 2-saturated paratopological group (G,τ) the maximal antidiscrete subgroup $\overline{\{e\}}=\bigcap_{e\in U\in\tau}(\pm U)^4$ of the topological group (G,τ_4) is nowhere dense in the topology τ_2.*

Proof. To show that $\overline{\{e\}}$ is nowhere dense in the topology τ_2, fix any non-empty open set $U_2\in\tau_2$. Since G is not discrete, so is the topology $\tau_2\subset\tau$. Consequently, we can find a point $x\in U_2\setminus\{e\}$. Since G is a Hausdorff paratopological group, there is a neighborhood $U\in\tau$ of e such that $e\notin xUU^{-1}\subset U_2$. The continuity of the group operation yields a neighborhood $V\in\tau$ of e such that $V^2\subset U$ and $V^2x\subset xU$. Then $V^2xV^{-2}\subset xUU^{-1}\not\ni e$ yields $V^{-1}V\cap VxV^{-1}=\varnothing$. Using the shift-invariantness of the topology τ, find a neighborhood $W\in\tau$ of e such that $W\subset V$ and $xW\subset Vx$.

Since the group G is 2-saturated, the open set $xWW^{-1}\in\tau_2$ has non-empty interior in the topology $(\tau^{-1})_2$. Consequently, there is a point $y\in xWW^{-1}$ and a neighborhood $O\in\tau$ of e such that $O\subset W$ and $O^{-1}yO\subset xWW^{-1}$. Observe that

$$O^{-1}O\cap O^{-1}yO\subset V^{-1}V\cap xWW^{-1}\subset V^{-1}V\cap VxV^{-1}=\varnothing$$

and consequently, $U_2\ni y\notin OO^{-1}OO^{-1}\supset\overline{\{e\}}$. □

Problem 3.4. *Can the topology τ_{2n} be antidiscrete for some Hausdorff n-saturated paratopological group?*

Problem 3.5. *Assume that a paratopological group (G,τ) is 2-saturated. Is its mirror paratopological group (G,τ^{-1}) 2-saturated?*

4. Proof of Theorem 1.1

We need to check that each meager Hausdorff paratopological group G is left and right meager.

If the paratopological group G contains a non-empty open left P-small subset, then G is left meager by Theorem 2.1(1). So assume that no non-empty open subset of G is left P-small. In this case Proposition 3.1 implies that the paratopological group G is 2-saturated while Theorem 3.3 ensures that the topological group (G,τ_4) contains a countable disjoint family $\{W_n\}_{n\in\omega}$ of non-empty open sets. By the definition of the 4th oscillator topology τ_4 each set W_n contains a subset of the form $x_nU_nU^{-1}U_nU_n^{-1}$ where $x_n\in G$ and U_n is a neighborhood of the neutral element in the paratopological group G. Since the sets $x_nU_nU_n^{-1}\subset W_n$, $n\in\omega$, are pairwise disjoint, we can apply Theorem 2.1(3) to conclude that the paratopological group G is left meager.

By analogy we can prove that G is right meager.

5. Discussion and Open Problems

The following example shows that without any restrictions, a meager semitopological group needs not be shift-meager.

Example 5.1. *Let G be an uncountable group whose cardinality $|G|$ has countable cofinality. Endow the group G with the shift-invariant topology generated by the base $\{G\setminus A : |A|<|G|\}$. It is easy to see that a subset $A\subset G$ is nowhere dense if and only if it is not dense if and only if $|A|<|G|$. This observation implies that G is meager (because $|G|$ has countable cofinality). On the other hand, the semi-topological group G is not shift-meager because for every nowhere dense subset $A\subset G$ and every countable subset $C\subset G$ we get $|A|<|G|$ and hence $G\neq CA$ because $|CA|\le\max\{\aleph_0,|A|\}<|G|$.*

Problem 5.2. *Is each meager paratopological group G shift-meager?*

Problem 5.3. *Is each meager Hausdorff semitopological group shift-meager?*

Problem 5.4. *Is each left meager semitopological group right meager?*

Also we do not know is the following semigroup version of Theorem 1.1 holds.

Problem 5.5. *Let S be an open meager subsemigroup of a Hausdorff paratopological group G. Is $S\subset CA$ for some nowhere dense subset $A\subset S$ and a countable subset $C\subset G$?*

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