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Hypercyclic abelian semigroup of matrices on $\mathbb{C}^n$ and $\mathbb{R}^n$ and k-transitivity ($k \geq 2$)

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Abstract

We prove that the minimal number of matrices on $\mathbb{C}^n$ required to form a hypercyclic abelian semigroup on $\mathbb{C}^n$ is $n+1$. We also prove that the action of any abelian semigroup finitely generated by matrices on $\mathbb{C}^n$ or $\mathbb{R}^n$ is never k-transitive for $k \geq 2$. These answer questions raised by Feldman and Javaheri.

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1. Introduction

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Following Feldman from [6], by an p-tuple of matrices, we mean a finite sequence of length p ($p \geq 1$) of commuting matrices $A_1, A_2, \dots, A_p$ on $\mathbb{K}^n$. We will let $G = \{A_1^{k_1} A_2^{k_2} \dots A_p^{k_p} : \ k_1, k_2, \dots, k_p \in \mathbb{N}\}$ be the *semigroup* generated by $A_1, A_2, \dots, A_p$. For a vector $x \in \mathbb{K}^n$, the orbit of x under the action of G on $\mathbb{K}^n$ is $\mathrm{O}_G(x) = \{Ax : \ A \in G\}$. For a subset $E \subset \mathbb{K}^n$, denote by $\overline{E}$ (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E. A subset $E \subset \mathbb{K}^n$ is called G-*invariant* if $A(E) \subset E$ for any $A \in G$. The orbit $\mathrm{O}_G(x) \subset \mathbb{K}^n$ is *dense* (resp. *locally dense*) in $\mathbb{K}^n$ if $\overline{O_G(x)} = \mathbb{K}^n$ (resp. $\overset{\circ}{\overline{O_G(x)}} \neq \varnothing$). The semigroup G is called *hypercyclic* (or also topologically transitive) (resp. *locally hypercyclic*) if there exists a vector $x \in \mathbb{K}^n$ such that $\mathrm{O}_G(x)$ is dense (resp. locally dense) in

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$\mathbb{K}^n$. For an account of results and bibliography on hypercyclicity, we refer to the book [3] by Bayart and Matheron.

On the other part, let $k \geq 1$ be an integer. Denote by $(\mathbb{K}^n)^k$ the k-fold Cartesian product of $\mathbb{K}^n$. For every $u = (x_1, \dots, x_k) \in (\mathbb{K}^n)^k$, the orbit of u under the action of G on $(\mathbb{K}^n)^k$ is denoted

$$O_G^k(u) = \{(Ax_1, \dots, Ax_k) : A \in G\}.$$

When $k = 1$, $\mathrm{O}_G^k(u) = O_G(u)$. We say that the action of G on $\mathbb{K}^n$ is k-*transitive* if, the induced action of G on $(\mathbb{K}^n)^k$ is hypercyclic, this is equivalent to that for some $u \in (\mathbb{K}^n)^k$, $\overline{O_G^k(u)} = (\mathbb{K}^n)^k$. A 2-transitive action is also called weak topological mixing and 1-transitive means hypercyclic.

In [6], Feldman showed that in $\mathbb{C}^n$ there exist a hypercyclic semigroup generated by $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$ and that no semigroup generated by n-tuple of diagonalizable matrices on $\mathbb{K}^n$ can be hypercyclic. If one remove the diagonalizability condition, Costakis et al. proved in [4] that there exists a hypercyclic semigroup generated by n-tuple of non diagonalizable matrices on $\mathbb{R}^n$. However, they show in [5] that there exist a hypercyclic semigroup generated by $(n+1)$-tuple of diagonalizable matrices $A_1, \dots, A_{n+1}$ on $\mathbb{R}^n$.

The main purpose of this paper is twofold: firstly, we give a general result (with respect to the results above) by showing that the minimal number of matrices on $\mathbb{C}^n$ required to form a hypercyclic tuple in $\mathbb{C}^n$ is $n+1$. This answer a question raised by Feldman in ([6], Section 6). Secondly, we prove that the action of any abelian semigroup finitely generated by matrices on $\mathbb{K}^n$ is never k-transitive for $k \geq 2$. This answer a question of Javaheri in ([7], Problem 3).

Our principal results are the following:

Theorem 1.1. *For every $n \geq 1$, any abelian semigroup generated by n matrices on $\mathbb{C}^n$ is not locally hypercyclic.*

Theorem 1.2. *Let G be an abelian semigroup generated by p matrices ($p \geq 1$) on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then the action of G on $\mathbb{K}^n$ is never k-transitive for $k \geq 2$.*

2. On hyercyclic semigroups

Let $\mathrm{M}_n(\mathbb{K})$ be the set of all square matrices of order $n \geq 1$ with entries in $\mathbb{K}$ and $\mathrm{GL}(n, \mathbb{K})$ be the group of invertible matrices of $\mathrm{M}_n(\mathbb{K})$. Let G be an abelian semigroup generated by p matrices ($p \geq 1$) on $\mathbb{K}^n$ and we let $G' = G \cap \mathrm{GL}(n, \mathbb{K})$.

Lemma 2.1. *Under the notation above, let $k \geq 1$ be an integer and $u \in (\mathbb{K}^n)^k$. Then*

(i) $\overline{O^k_G(u)} = (\mathbb{K}^n)^k$ *if and only if* $\overline{O^k_{G'}(u)} = (\mathbb{K}^n)^k$.

(ii) $\overset{\circ}{\overline{O^k_G(u)}} = \varnothing$ *if and only if* $\overset{\circ}{\overline{O^k_{G'}(u)}} = \varnothing$.

In particular, if the action of G on $\mathbb{K}^n$ is k-transitive so is the action of G' on $\mathbb{K}^n$.

Proof. (i) Suppose that $\overline{O^k_{G'}(u)} = (\mathbb{K}^n)^k$ for some $u \in (\mathbb{K}^n)^k$. Then since $\overline{O^k_{G'}(u)} \subset \overline{O^k_G(u)}$, we see that $\overline{O^k_G(u)} = (\mathbb{K}^n)^k$.

Conversely, suppose there exists $u \in (\mathbb{K}^n)^k$ such that $\overline{O^k_G(u)} = (\mathbb{K}^n)^k$. Denote by $(A_1, \dots, A_p)$ an p-tuple of matrices on $\mathbb{K}^n$ which generate the semigroup G. One can suppose that for some $0 \leq r \leq p$, $A_1, \dots, A_r \in \mathrm{GL}(n, \mathbb{K})$ and $A_{r+1}, \dots, A_p \in M_n(\mathbb{K}) \backslash \mathrm{GL}(n, \mathbb{K})$. Then $G' = G \cap \mathrm{GL}(n, \mathbb{K})$ is the semigroup generated by $A_1, \dots, A_r$. For $k = 1, \dots, r$, write $\mathrm{Im}(A_k) = A_k(\mathbb{K}^n)$ the range of A_k. Then $\mathrm{Im}(A_k)$ is a vector subspace of $\mathbb{K}^n$ of dimension $< n$, hence $\overset{\circ}{\mathrm{Im}(A_k)} = \varnothing$.

- If $r = p$ then $G = G'$ and so (i) is obvious.
- If $r = 0$ then for every $u \in (\mathbb{K}^n)^k$, $\mathrm{O}^k_G(u) \subset \bigcup\limits_{k=1}^{p} (\mathrm{Im}(A_k))^k \cup \{u\}$. Since $\overset{\circ}{\bigcup\limits_{k=1}^{p} (\mathrm{Im}(A_k))^k} = \varnothing$, $\overset{\circ}{\overline{O^k_G(u)}} = \varnothing$.
- If $0 < r < p$ then

$$O^k_G(u) \subset \left(\bigcup_{j=1}^{r} (\mathrm{Im}(A_j))^k \right) \cup O^k_{G'}(u).$$

It follows that

$$(\mathbb{K}^n)^k \subset \left(\bigcup_{j=1}^{r} (\mathrm{Im}(A_j))^k \right) \cup \overline{O^k_{G'}(u)}$$

and therefore $\overline{O^k_{G'}(u)} = (\mathbb{K}^n)^k$.

The proof of (ii) is the same as for (i). □

Lemma 2.2 ([2], Corollary 1.5)**.** *Let G be an abelian subgroup of $GL(n, \mathbb{C})$. If G is generated by n matrices ($n \geq 1$), it has no dense orbit.*

Lemma 2.3 ([6], Corollary 5.7)**.** *Let G be an abelian semigroup generated by p matrices ($p \geq 1$) on $\mathbb{C}^n$. Then every locally dense orbit of G is dense in $\mathbb{C}^n$.*

From Lemmas 2.2 and 2.3, we obtain the following:

Corollary 2.4. *Any abelian semigroup generated by n matrices ($n \geq 1$) of $GL(n, \mathbb{C})$ is not locally hypercyclic.*

Proof of Theorem 1.1. Let G be an abelian semigroup generated by n matrices on $\mathbb{C}^n$ and we let $G' = G \cap \mathrm{GL}(n, \mathbb{C})$. By Corollary 2.4, $\overset{\circ}{\overline{O^k_{G'}(u)}} = \varnothing$ for every $u \in (\mathbb{C}^n)^k$ and hence by Lemma 2.1, $\overset{\circ}{\overline{O^k_G(u)}} = \varnothing$. The proof is complete. □

3. On k-transitivity $(k \geq 2)$

Let recall first the following result:

Proposition 3.1 ([1], Theorem 4.1)**.** *Let G be an abelian subgroup of $GL(n, \mathbb{K})$ $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$. Then there exists a G-invariant dense open subset U in $\mathbb{K}^n$ such that if, $u, v \in U$ and $(B_m)_{m\in\mathbb{N}}$ is a sequence of G such that $\lim\limits_{m\to+\infty} B_m u = v$ then $\lim\limits_{m\to+\infty} B_m^{-1} v = u$.*

Corollary 3.2. *Let G be an abelian subgroup of $GL(n, \mathbb{K})$ $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$ and let U be a G-invariant dense open subset of $\mathbb{K}^n$ as in Proposition 3.1. Then for every $k \geq 2$, if $v \in U^k$ and $w \in \overline{O^k_G(v)} \cap U^k$ then $\overline{O^k_G(v)} \cap U^k = \overline{O^k_G(w)} \cap U^k$.*

Proof. Write $v = (v_1, \dots, v_k)$, $w = (w_1, \dots, w_k) \in U^k$. Suppose that $w \in \overline{O^k_G(v)} \cap U^k$. Then there exists a sequence $(B_m)_{m\in\mathbb{N}}$ in G such that

$$\lim_{m\to+\infty} (B_m v_1, \dots, B_m v_k) = (w_1, \dots, w_k).$$

Then $\lim\limits_{m\to+\infty} B_m v_j = w_j$, for every $1 \leq j \leq k$. Since v_j, $w_j \in U$, so by Proposition 3.1, $\lim\limits_{m\to+\infty} B_m^{-1} w_j = v_j$ and hence

$$\lim_{m\to+\infty} (B_m^{-1} w_1, \dots, B_m^{-1} w_k) = v \in \overline{O^k_G(w)}.$$

It follows that $\overline{O^k_G(v)} \cap U^k = \overline{O^k_G(w)} \cap U^k$. □

Proof of Theorem 1.2. Suppose the action of G is k-transitive $(k \geq 2)$, then there exists $v = (v_1, \dots, v_k) \in (\mathbb{K}^n)^k$ so that $\overline{O^k_G(v)} = (\mathbb{K}^n)^k$. We let $G' = G \cap \mathrm{GL}(n, \mathbb{K})$. By Lemma 2.1, $\overline{O^k_{G'}(v)} = (\mathbb{K}^n)^k$. Denote by G'' the group generated by G' and by U a G''-invariant dense open subset in $\mathbb{K}^n$ as in Proposition 3.1. Then $\overline{O^k_{G''}(v)} = (\mathbb{K}^n)^k$ and hence $v \in U^k$. Write

$w := (v_1, \dots, v_1)$. Then $w \in U^k$ and by Corollary 3.2, $\overline{O^k_{G''}(w)} = (\mathbb{K}^n)^k$ (since U^k is dense in $(\mathbb{K}^n)^k$). It follows that $\overline{O_{G''}(v_1)} = \mathbb{K}^n$.

Let $\varphi : \mathbb{K}^n \longrightarrow (\mathbb{K}^n)^k$ be the homomorphism defined by

$$\varphi(x) = (x, \dots, x), \; x \in \mathbb{K}^n.$$

Then $\mathrm{O}^k_{G''}(w) = \varphi(O_{G''}(v_1)) \subset \varphi(\mathbb{K}^n)$. As $\varphi(\mathbb{K}^n)$ is a vector subspace of $(\mathbb{K}^n)^k$ of dimension $n < nk$, $\mathrm{O}_{G''}(w)$ cannot be dense in $(\mathbb{K}^n)^k$ (since $k \geq 2$), this is a contradiction and the theorem is proved. □

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