

## Bounded point evaluations for cyclic Hilbert space operators

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Dedicated to Professor S. Naimpally on the occasion of his 70<sup>th</sup> birthday.

**ABSTRACT.** In this talk, to be given at a conference at Seconda Università degli Studi di Napoli in September 2001, we shall describe the set of analytic bounded point evaluations for an arbitrary cyclic bounded linear operator  $T$  on a Hilbert space  $\mathcal{H}$  and shall answer some questions due to L. R. Williams.

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### 1. INTRODUCTION.

Throughout this paper,  $\mathcal{L}(\mathcal{H})$  will denote the algebra of all linear bounded operators on an infinite-dimensional separable complex Hilbert space  $\mathcal{H}$ . Let  $T \in \mathcal{L}(\mathcal{H})$  be a cyclic operator on  $\mathcal{H}$  with cyclic vector  $x \in \mathcal{H}$  i.e., the linear subspace  $\{p(T)x : p \text{ polynomial}\}$  is dense in  $\mathcal{H}$ . A complex number  $\lambda \in \mathbb{C}$  is said to be a *bounded point evaluation* for  $T$  if there is a positive constant  $M$  such that for every polynomial  $p$ ,

$$|p(\lambda)| \leq M\|p(T)x\|;$$

equivalently, if  $\lambda$  induces a continuous linear functional on  $\mathcal{H}$  which maps  $p(T)x$  to  $p(\lambda)$  for every polynomial  $p$ . Therefore, it follows from the Riesz Representation Theorem that a complex number  $\lambda \in \mathbb{C}$  is a bounded point evaluation for  $T$  if and only if there is a unique vector  $k(\lambda) \in \mathcal{H}$  such that

$$(1.1) \quad p(\lambda) = \langle p(T)x, k(\lambda) \rangle \text{ for every polynomial } p.$$

The set of all bounded point evaluations for  $T$  will be denoted by  $B(T)$ . A point  $\lambda \in B(T)$  is called an *analytic bounded point evaluation* for  $T$  if there is

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an open neighborhood  $O$  of  $\lambda$  contained in  $B(T)$  such that for every  $y \in \mathcal{H}$ , the complex function  $\widehat{y}$  defined on  $B(T)$  by  $\widehat{y}(\lambda) := \langle y, k(\lambda) \rangle$ , is analytic on  $O$ . The set of all analytic bounded point evaluations for  $T$  will be denoted by  $B_a(T)$ .

An operator  $T \in \mathcal{L}(\mathcal{H})$  is said to be *subnormal* if it has a normal extension, i.e., if there is a normal operator  $N$  on a Hilbert space  $\mathcal{K}$ , containing  $\mathcal{H}$ , such that  $\mathcal{H}$  is a closed invariant subspace of  $N$  and the restriction  $N|_{\mathcal{H}}$  coincides with  $T$ . The operator  $T$  is said to be *hyponormal* if  $\|T^*x\| \leq \|Tx\|$  for every  $x \in \mathcal{H}$ , where  $T^*$  denotes the adjoint of  $T$ . Note that every subnormal operator is hyponormal, with the converse false (see [8] and also Example 5.6). Using Bran's theorem [5] and the maximum modulus principle for analytic functions, Tavan T. Trent proved in [20] that for every cyclic subnormal operator  $T \in \mathcal{L}(\mathcal{H})$ , we have

$$B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T).$$

Here,  $\sigma_{ap}(T)$  denotes the approximate point spectrum of  $T$ , that is the set of complex numbers  $\lambda$  for which there is a sequence  $(x_n)_n$  of elements of the unit sphere of  $\mathcal{H}$  such that  $\lim_{n \rightarrow +\infty} \|(T - \lambda)x_n\| = 0$ , and  $\Gamma(T)$  denotes the compression spectrum of  $T$ , that is the set of complex numbers  $\lambda$  such that the range of  $(T - \lambda)$  is not dense in  $\mathcal{H}$ . More informations about bounded point evaluations for cyclic subnormal operators can be found in [8].

In [21], L. R. Williams followed Trent's method to shown that

$$(1.2) \quad \Gamma(T) \setminus \sigma_{ap}(T) \subset B_a(T)$$

for every cyclic operator  $T \in \mathcal{L}(\mathcal{H})$  and posed the following question.

**Question 1.1.** *Let  $T \in \mathcal{L}(\mathcal{H})$  be a cyclic operator. Is  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$ ?*

Note that, in general, the basic spectral properties of subnormal operators remain valid for hyponormal operators. Thus we pose the following weaker question.

**Question 1.2.** *Is  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$  for every cyclic hyponormal operator  $T \in \mathcal{L}(\mathcal{H})$ ?*

In this paper, we shall explain more about bounded point evaluations for cyclic Hilbert space operators from the point of view of local spectral theory and shall answer the above questions. In Section 2, we give a complete description of the set of analytic bounded point evaluations for arbitrary cyclic operators and derive some consequences from it. In Section 3, we give a necessary and sufficient condition for unilateral weighted shift operators to satisfy Trent's result, and exhibit some operators which provide a negative answer to Question 1.1. In Section 4, we generalize a result of L. Yang which allows us to give a positive answer to Question 1.2. As a corollary, we get that two quasimilar cyclic hyponormal operators have equal approximate point spectra; this result is a generalization of Theorem 4 of [16]. In Section 5, we show that if  $T \in \mathcal{L}(\mathcal{H})$  is a cyclic operator for which the span of the eigenvectors of  $T^*$  associated with

a connected component of  $B_a(T)$  is dense in  $\mathcal{H}$ , then  $T$  is without eigenvalues and has a connected spectrum. Some related examples are given.

Before going further, we need to introduce some notations and recall some basic notions concerning local spectral theory; we refer to the monographs [7] and [13] for further informations. For an operator  $T \in \mathcal{L}(\mathcal{H})$ , we denote as usual by  $\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\}$ ,  $\rho(T) := \mathbb{C} \setminus \sigma(T)$ ,  $\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not injective}\}$ ,  $\ker T$ , and  $\text{ran} T$  the spectrum, the resolvent set, the point spectrum, the kernel, and the range of  $T$ , respectively. For a subset  $M$  of  $\mathcal{H}$ , we use  $\text{cl}(M)$ , and  $\bigvee M$  to denote the closure of  $M$ , and the closed linear subspace generated by  $M$ , respectively. For a subset  $F$  of  $\mathbb{C}$ , we use  $\overline{F}$ , and  $\text{Fr}(F)$  to denote the complex conjugates of the points in  $F$ , and the boundary of  $F$ , respectively. For an open subset  $U$  of  $\mathbb{C}$ , we let  $\mathcal{O}(U, \mathcal{H})$  denote the space of analytic  $\mathcal{H}$ -valued functions on  $U$ . It is a Fréchet space when equipped with the topology of uniform convergence on compact subsets of  $U$  and the space  $\mathcal{H}$  may be viewed as simply the constants in  $\mathcal{O}(U, \mathcal{H})$ . One says that an operator  $T \in \mathcal{L}(\mathcal{H})$  possesses *Bishop's property* ( $\beta$ ) if for each open subset  $U$  of  $\mathbb{C}$ , the multiplication operator

$$T_U : \mathcal{O}(U, \mathcal{H}) \longrightarrow \mathcal{O}(U, \mathcal{H}), \quad f \longmapsto (T - z)f$$

is injective with closed range. M. Putinar [14] has shown that hyponormal operators have Bishop's property ( $\beta$ ). Recall that  $T$  is said to have the *single-valued extension property* provided that, for every open subset  $U$  of  $\mathbb{C}$ , the only analytic  $\mathcal{H}$ -valued solution  $f$  of the equation  $(T - \lambda)f(\lambda) = 0$ , ( $\lambda \in U$ ), is the identically zero function  $f \equiv 0$  on  $U$ . Equivalently, if for every open subset  $U$  of  $\mathbb{C}$ , the mapping  $T_U$  is injective. A localized version of this property dates back to J. K. Finch [10] and can be defined as follows (see [1], and [10]). An operator  $T \in \mathcal{L}(\mathcal{H})$  is said to have the *single-valued extension property* at a point  $\lambda \in \mathbb{C}$  if for every open disk  $U$  centered at  $\lambda$ , the mapping  $T_U$  is injective. Let

$$\mathfrak{R}(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the single-valued extension property at } \lambda\}.$$

Obviously,  $\mathfrak{R}(T)$  is an open subset contained in  $\sigma_p(T)$  and is empty precisely when  $T$  has the single-valued extension property. The *local resolvent set* of  $T$  at a vector  $y \in \mathcal{H}$ , denoted by  $\rho_T(y)$ , is the union of all open subsets  $U$  of  $\mathbb{C}$  for which  $y \in \text{ran} T_U$ . The *local spectrum* of  $T$  at  $y$  is  $\sigma_T(y) := \mathbb{C} \setminus \rho_T(y)$ ; it is a closed subset of  $\sigma(T)$ .

In the sequel,  $T \in \mathcal{L}(\mathcal{H})$  will be a cyclic operator with cyclic vector  $x \in \mathcal{H}$ ; and for  $\lambda \in B(T)$ ,  $k(\lambda)$  will denote the vector of  $\mathcal{H}$  as defined in (1.1).

## 2. DESCRIPTION OF $B(T)$ AND $B_a(T)$ .

The proofs of Proposition 2.1 and Lemma 2.2 are similar to the ones for the cyclic subnormal operators (see [2], and [8]); we include them for completeness. A complete description of  $B(T)$  is given by the following result.

**Proposition 2.1.** *Let  $\lambda \in \mathbb{C}$ ; the following statements are equivalent.*

- (i)  $\lambda \in B(T)$ .
- (ii)  $\lambda \in \Gamma(T)$ .
- (iii)  $\ker((T - \lambda)^*)$  is one dimensional.

*Proof.* First, note that if  $(T - \lambda)^*u = 0$  for some  $u \in \mathcal{H}$ , then for every polynomial  $p$ , we have

$$(2.3) \quad \langle p(T)x, u \rangle = p(\lambda)\langle x, u \rangle.$$

Next, we mention that it suffices to prove the implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii), since the implication (iii) $\Rightarrow$ (ii) can be deduced trivially from the fact that  $\Gamma(T) = \overline{\sigma_p(T^*)}$ .

Let  $\lambda \in B(T)$ ; it is clear that  $k(\lambda) \neq 0$  since  $\langle x, k(\lambda) \rangle = 1$ . On the other hand, for every polynomial  $p$ , we have

$$\begin{aligned} 0 &= \langle (T - \lambda)p(T)x, k(\lambda) \rangle \\ &= \langle p(T)x, (T - \lambda)^*k(\lambda) \rangle. \end{aligned}$$

Since  $x$  is a cyclic vector for  $T$ , we have  $(T - \lambda)^*k(\lambda) = 0$ . Hence,  $\bar{\lambda} \in \sigma_p(T^*)$ . Now, let  $u \in \mathcal{H}$  be such that  $(T - \lambda)^*u = 0$ . It follows then from equation (2.3) that  $u = \overline{\langle x, u \rangle}k(\lambda)$ . Therefore, (i) $\Rightarrow$ (iii).

Let  $\lambda \in \Gamma(T)$ . Then there is a non-zero vector  $u \in \mathcal{H}$  such that  $(T - \lambda)^*u = 0$ . Since  $x$  is a cyclic vector of  $T$ , it follows from equation (2.3) that  $\langle x, u \rangle \neq 0$ . Hence,

$$p(\lambda) = \langle p(T)x, \frac{u}{\langle x, u \rangle} \rangle \text{ for every polynomial } p.$$

Therefore,  $\lambda \in B(T)$  and  $k(\lambda) = \frac{u}{\langle x, u \rangle}$ , which proves that (ii) $\Rightarrow$ (i).  $\square$

**Lemma 2.2.** *An open subset  $O$  of  $\mathbb{C}$  is contained in  $B_a(T)$  if and only if it is contained in  $B(T)$  and the function  $\lambda \mapsto \|k(\lambda)\|$  is bounded on compact subsets of  $O$ .*

*Proof.* Assume that  $O \subset B_a(T)$  and let  $K$  be a compact subset of  $O$ . For every  $y \in \mathcal{H}$ , the function  $\hat{y}$  is analytic on  $O$ ; in particular,  $\sup_{\lambda \in K} |\langle y, k(\lambda) \rangle| < +\infty$ . So it follows from the Uniform Boundedness Principle that

$$\sup_{\lambda \in K} \|k(\lambda)\| < +\infty.$$

Conversely, suppose that  $O \subset B(T)$  and the function  $\lambda \mapsto \|k(\lambda)\|$  is bounded on compact subsets of  $O$ . Let  $y \in \mathcal{H}$ ; then there is a sequence of polynomials  $(p_n)_n$  such that  $\lim_{n \rightarrow +\infty} \|p_n(T)x - y\| = 0$ . And so, for every compact subset  $K$  of  $O$ , it follows from the Cauchy-Schwartz inequality that,

$$\sup_{\lambda \in K} |p_n(\lambda) - \hat{y}(\lambda)| \leq \sup_{\lambda \in K} \|k(\lambda)\| \|p_n(T)x - y\|.$$

Hence, the function  $\hat{y}$  is analytic on  $O$ . Therefore,  $O \subset B_a(T)$ .  $\square$

The following gives a complete description of  $B_a(T)$ .

**Theorem 2.3.**  $B_a(T) = \overline{\mathfrak{R}(T^*)}$ .

*Proof.* Suppose that  $\lambda \in \mathfrak{R}(T^*)$ . Then there is a non-zero analytic  $\mathcal{H}$ -valued function  $\phi$  on some open disk  $\mathcal{V}$  centered at  $\lambda$  such that

$$(T - \mu)\phi(\mu) = 0 \text{ for all } \mu \in \mathcal{V}.$$

Using the fact that a non-zero analytic  $\mathcal{H}$ -valued function has isolated zeros, one can assume that the function  $\phi$  has no zero in  $\mathcal{V}$ . Hence,  $\mathcal{V} \subset \sigma_p(T^*) = \overline{B(T)}$ . As before, it follows from (2.3) that

$$k(\bar{\mu}) = \frac{\phi(\mu)}{\langle x, \phi(\mu) \rangle} \text{ for every } \mu \in \mathcal{V}.$$

Therefore, the function  $k : \overline{\mathcal{V}} \rightarrow \mathcal{H}$  is continuous; in particular, the function  $\mu \mapsto \|k(\mu)\|$  is bounded on compact subsets of  $\overline{\mathcal{V}}$ . By Lemma 2.2,  $\overline{\mathcal{V}} \subset B_a(T)$ . Thus  $\mathfrak{R}(T^*) \subset B_a(T)$ .

Conversely, set  $O = \overline{B_a(T)}$  and consider the following  $\mathcal{H}$ -valued function  $\phi$  defined on  $O$  by

$$\phi(\lambda) := k(\bar{\lambda}), \lambda \in O.$$

First, we show that  $\phi$  is analytic on  $O$ . Indeed, for every  $y \in \mathcal{H}$  and for every  $\lambda_0 \in O$ , we have

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_0} \frac{\langle \phi(\lambda), y \rangle - \langle \phi(\lambda_0), y \rangle}{\lambda - \lambda_0} &= \lim_{\lambda \rightarrow \lambda_0} \frac{\langle k(\bar{\lambda}), y \rangle - \langle k(\bar{\lambda}_0), y \rangle}{\lambda - \lambda_0} \\ &= \lim_{\lambda \rightarrow \lambda_0} \frac{\widehat{y}(\bar{\lambda}) - \widehat{y}(\bar{\lambda}_0)}{\lambda - \lambda_0} \\ &= \widehat{y}(\bar{\lambda}_0). \end{aligned}$$

Hence, for every  $y \in \mathcal{H}$ , the function  $\lambda \mapsto \langle \phi(\lambda), y \rangle$  is differentiable on  $O$ ; therefore,  $\phi$  is analytic on  $O$ . On the other hand,  $\phi$  has no zeros on  $O$  and satisfies the following equation

$$(T^* - \lambda)\phi(\lambda) = 0 \text{ for every } \lambda \in O.$$

This gives  $O = \overline{B_a(T)} \subset \mathfrak{R}(T^*)$ , and the proof is complete. □

**Corollary 2.4.** *The following holds:*

$$\begin{aligned} B_a(T) &= \{\lambda \in \Gamma(T) : \sigma_{T^* - \bar{\lambda}}(k(\lambda)) = \emptyset\} \\ &= \{\lambda \in \Gamma(T) : \sigma_{T^*}(k(\lambda)) = \emptyset\}. \end{aligned}$$

*Proof.* Since, for every  $\lambda \in B_a(T)$ ,  $\bar{\lambda}$  is a simple eigenvalue of  $T^*$  with corresponding eigenvector  $k(\lambda)$ , the proof follows by combining Theorem 2.3 and Theorem 1.9 of [1]. □

**Remark 2.5.** (i) Note that Proposition 2.1 and Proposition 2.3 show that both  $B(T)$  and  $B_a(T)$  are independent of the choice of cyclic vector for  $T$  (see Proposition 1.4 of [21]).

(ii) Using Theorem 2.3 and Theorem 2.6 of [1], one can easily prove (1.2).

## 3. RESOLUTION OF QUESTION 1.1.

The weighted shift operators have proven to be an interesting rich collection of operators providing examples and counterexamples to illustrate many properties of operators. The Allen Shields's excellent survey [18] contains their basic facts and properties concerning their spectral theory (see also [2]).

Throughout this section,  $S$  will denote a unilateral weighted shift operator on  $\mathcal{H}$  with a positive bounded weight sequence  $(\omega_n)_{n \geq 0}$ , that is

$$S e_n = \omega_n e_{n+1}, \quad n \geq 0,$$

where  $(e_n)_{n \geq 0}$  is an orthonormal basis of  $\mathcal{H}$ . Let  $(\beta_n)_{n \geq 0}$  be the following sequence given by:

$$\beta_n = \begin{cases} \omega_0 \dots \omega_{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

Set

$$r_1(S) = \lim_{n \rightarrow \infty} \left[ \inf_{k \geq 0} \frac{\beta_{n+k}}{\beta_k} \right]^{\frac{1}{n}}, \quad r_2(S) = \liminf_{n \rightarrow \infty} [\beta_n]^{\frac{1}{n}} \quad \text{and} \quad r(S) = \lim_{n \rightarrow \infty} \left[ \sup_{k \geq 0} \frac{\beta_{n+k}}{\beta_k} \right]^{\frac{1}{n}};$$

and note that,

$$r_1(S) \leq r_2(S) \leq r(S) \leq \|S\|.$$

The following gives a necessary and sufficient condition for the weighted shift  $S$  to answer affirmatively Question 1.1.

**Theorem 3.1.** *The following are equivalent.*

- (i)  $B_a(S) = \Gamma(S) \setminus \sigma_{ap}(S)$ .
- (ii)  $r_1(S) = r_2(S)$ .

*Proof.* Since  $\sigma_p(S^*) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r_2(S)\}$  (see Theorem 9 of [18]), we have

$$\Re(S^*) \subset O := \{\lambda \in \mathbb{C} : |\lambda| < r_2(S)\}.$$

Conversely, consider the following non-zero analytic  $\mathcal{H}$ -valued function  $\phi$  defined on  $O$  by

$$\phi(\lambda) := \sum_{n=0}^{+\infty} \frac{\lambda^n}{\beta_n} e_n.$$

It is easy to see that  $(S^* - \lambda)\phi(\lambda) = 0$  for every  $\lambda \in O$ . Hence,

$$O = \{\lambda \in \mathbb{C} : |\lambda| < r_2(S)\} \subset \Re(S^*).$$

Therefore,  $\Re(S^*) = \{\lambda \in \mathbb{C} : |\lambda| < r_2(S)\}$ ; by Theorem 2.3,

$$(3.4) \quad B_a(S) = \{\lambda \in \mathbb{C} : |\lambda| < r_2(S)\}.$$

On the other hand, by [18, Theorems 4 and 6], the spectrum and the approximate point spectrum of  $S$  are given, respectively, by

$$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(S)\} \quad \text{and} \quad \sigma_{ap}(S) = \{\lambda \in \mathbb{C} : r_1(S) \leq |\lambda| \leq r(S)\}.$$

Hence,

$$(3.5) \quad \Gamma(S) \setminus \sigma_{ap}(S) = \sigma(S) \setminus \sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| < r_1(S)\}.$$

And so the proof follows from (3.4) and (3.5). □

Now, to give a counterexample to Question 1.1, we only need to produce a weight sequence  $(\omega_n)_{n \geq 0}$  for which the corresponding weighted shift  $S$  satisfies  $r_1(S) < r_2(S)$ .

**Example 3.2.** Let  $(C_k)_{k \geq 0}$  be the sequence of successive disjoint segments covering the set of non-negative integers  $\mathbb{N}$  such that each segment  $C_k$  contains  $k^2 + k$  elements. Let  $R > 1$  be a real number and let  $k \in \mathbb{N}$ ; for every  $n \in C_k$  we set,

$$\omega_n = \begin{cases} R & \text{if } n \text{ lies in the first } k^2 \text{ terms of } C_k \\ 1 & \text{otherwise} \end{cases}$$

Hence, the unilateral weighted shift  $S$  corresponding to the weight  $(\omega_n)_{n \geq 0}$  is bounded and satisfies  $\|S\| = R$  and  $r_1(S) = 1$ . On the other hand for every  $n \geq 3$ , there is  $k(n) \geq 2$  such that  $n \in C_{k(n)}$ . Hence,

$$R \sum_{s=1}^{k(n)-1} s^2 \leq \beta_n \text{ and } n \leq \sum_{s=1}^{k(n)} (s^2 + s).$$

Therefore,

$$R^{\frac{(2k(n)-1)(k(n)-1)}{2(k(n)+1)(k(n)+2)}} \leq [\beta_n]^{\frac{1}{n}}.$$

Since  $\lim_{n \rightarrow +\infty} k(n) = +\infty$ , it follows that,

$$R \leq \liminf_{n \rightarrow +\infty} [\beta_n]^{\frac{1}{n}}.$$

We deduce that

$$r_1(S) = 1 \text{ and } r_2(S) = \|S\| = R.$$

Thus,

$$\Gamma(S) \setminus \sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \subsetneq B_a(S) = \{\lambda \in \mathbb{C} : |\lambda| < R\}.$$

The original idea of this construction is due to W. C. Ridge [17].

For other example see [3], where the authors constructed a unilateral weighted shift  $S$  for which  $\Gamma(S) \setminus \sigma_{ap}(S) = \emptyset$  and  $B_a(S) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ .

#### 4. RESOLUTION OF QUESTION 1.2.

If  $T$  possesses Bishop's property  $(\beta)$ , then we obtain the following.

**Theorem 4.1.** *If  $T$  possesses Bishop's property  $(\beta)$ , then the following are equivalent.*

- (i)  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$ .
- (ii)  $B_a(T) \cap \sigma_p(T) = \emptyset$ .

*Proof.* If  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$  then it is clear that  $B_a(T) \cap \sigma_p(T) = \emptyset$  since  $\sigma_p(T) \subset \sigma_{ap}(T)$ . Conversely, suppose that  $B_a(T) \cap \sigma_p(T) = \emptyset$ . Since  $\Gamma(T) \setminus \sigma_{ap}(T) \subset B_a(T) \subset \Gamma(T)$ , it suffices to prove that  $B_a(T) \cap \sigma_{ap}(T) = \emptyset$ . Suppose that there is  $\lambda \in B_a(T) \cap \sigma_{ap}(T)$ . It then follows that  $\text{ran}(T - \lambda)$  is not closed. Let  $y \in \text{cl}(\text{ran}(T - \lambda)) \setminus \text{ran}(T - \lambda)$ ; then

$$y \notin \text{ran}(T - \lambda) \text{ and } \langle y, k(\lambda) \rangle = 0.$$

Therefore, there is a sequence of polynomials  $(p_n)_n$  vanishing at  $\lambda$  such that  $\lim_{n \rightarrow +\infty} \|p_n(T)x - y\| = 0$ . Define on  $U := B_a(T)$  the following analytic  $\mathcal{H}$ -valued functions  $f$  and  $f_n$  by  $f(\mu) = y - \widehat{y}(\mu)x$  and  $f_n(\mu) = p_n(T)x - p_n(\mu)x$ ,  $n \geq 0$ . Since  $f(\lambda) = y \notin \text{ran}(T - \lambda)$ , then  $f \notin \text{ran}(T_U)$ . On the other hand, it is easy to see that  $f_n \in \text{ran}(T_U)$  for every  $n \geq 0$ . Now, let  $K$  be a compact subset of  $U$ ; we have

$$\begin{aligned} \sup_{\mu \in K} \|f_n(\mu) - f(\mu)\| &\leq \|p_n(T)x - y\| + \sup_{\mu \in K} \|[p_n(\mu) - \widehat{y}(\mu)]x\| \\ &\leq \|p_n(T)x - y\| + \|x\| \sup_{\mu \in K} |p_n(\mu) - \widehat{y}(\mu)| \\ &\leq \left[1 + \|x\| \sup_{\mu \in K} \|k(\mu)\|\right] \|p_n(T)x - y\|. \end{aligned}$$

Therefore,  $f_n \rightarrow f$  in  $\mathcal{O}(U, \mathcal{H})$ . Thus  $\text{ran}(T_U)$  is not closed which contradicts the fact that  $T$  possess Bishop's property  $(\beta)$ . The proof is complete.  $\square$

The following is immediate (see Theorem 3.1 of [23]).

**Corollary 4.2.** *Suppose that  $T$  possesses Bishop's property  $(\beta)$ . If  $\sigma_p(T) = \emptyset$ , then  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$ .*

The following gives a positive answer to Question 1.2.

**Theorem 4.3.** *If  $T$  is hyponormal, then  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$ .*

*Proof.* Since  $T$  is hyponormal, then for every  $\lambda \in \mathbb{C}$ , we have  $T - \lambda$  is hyponormal i.e.,

$$(4.6) \quad \|(T - \lambda)^* y\| \leq \|(T - \lambda)y\|, \quad \forall y \in \mathcal{H}.$$

Now, let  $\lambda \in B_a(T)$  and suppose that there is  $y \in \mathcal{H}$  such that  $Ty = \lambda y$ . For every  $\mu \in B_a(T)$ , we have,

$$\begin{aligned} \lambda \widehat{y}(\mu) &= \langle Ty, k(\mu) \rangle \\ &= \langle y, T^* k(\mu) \rangle \\ &= \langle y, \overline{\mu} k(\mu) \rangle \\ &= \mu \widehat{y}(\mu). \end{aligned}$$

Hence, the analytic function  $\widehat{y}$  is identically zero on  $B_a(T)$ . On the other hand, it follows from (4.6) and Proposition 2.1 that  $y = \alpha k(\lambda)$  for some  $\alpha \in \mathbb{C}$ . Therefore,  $\widehat{y}(\lambda) = \alpha \|k(\lambda)\|^2 = 0$  gives  $y = 0$ . Thus,  $B_a(T) \cap \sigma_p(T) = \emptyset$ . By Theorem 4.1, the proof is complete.  $\square$



In [16], M. Raphael has shown that two quasisimilar cyclic subnormal operators have equal approximate point spectra. The following generalizes M. Raphael's result to cyclic hyponormal operators. Suppose that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are Hilbert spaces. Recall that two operators  $T_1 \in \mathcal{L}(\mathcal{H}_1)$  and  $T_2 \in \mathcal{L}(\mathcal{H}_2)$  are said to be *quasisimilar* if there exist two bounded linear transformations  $X : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  and  $Y : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  injectives and having dense range such that  $XT_1 = T_2X$  and  $T_1Y = YT_2$ .

**Corollary 4.4.** *Suppose that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are Hilbert spaces, and let  $T_1 \in \mathcal{L}(\mathcal{H}_1)$  and  $T_2 \in \mathcal{L}(\mathcal{H}_2)$ . If  $T_1$  and  $T_2$  are quasisimilar cyclic hyponormal operators, then  $\sigma_{ap}(T_1) = \sigma_{ap}(T_2)$ .*

*Proof.* In view of Theorem 1.5 of [21] and Theorem 4.3, we have

$$\Gamma(T_1) \setminus \sigma_{ap}(T_1) = \Gamma(T_2) \setminus \sigma_{ap}(T_2).$$

Hence,

$$\sigma(T_1) \setminus \sigma_{ap}(T_1) = \sigma(T_2) \setminus \sigma_{ap}(T_2).$$

Since  $\sigma(T_1) = \sigma(T_2)$  (see Theorem 2 of [6], and also Corollary 2.2 of [23]), we deduce that  $\sigma_{ap}(T_1) = \sigma_{ap}(T_2)$ .  $\square$

**Remark 4.5.** Theorem 4.3 and Corollary 4.4 can be extended with no extra effort to the class of operators satisfying the following.

- (i)  $T$  possesses Bishop's property  $(\beta)$ .
- (ii)  $\ker(T - \lambda) \subset \ker(T - \lambda)^*$  for every  $\lambda \in \mathbb{C}$ .

Immediate other examples of operators satisfying (i) and (ii) are provided by  $M$ -hyponormal and  $p$ -hyponormal operators (see [14] and [9]).

### 5. EXAMPLES AND COMMENTS.

In this section, we start by proving the following result that we will need throughout.

**Proposition 5.1.** *If  $\mathcal{H} = \bigvee \{k(\lambda) : \lambda \in B_a(T)\}$ , then  $\sigma_p(T) = \emptyset$ . Moreover, if  $\mathcal{H} = \bigvee \{k(\lambda) : \lambda \in G\}$  for some connected component  $G$  of  $B_a(T)$ , then the following hold.*

- (i)  $\text{cl}(G) \subset \sigma_T(y)$  for every non-zero element  $y \in \mathcal{H}$ .
- (ii) For every  $y \in \mathcal{H}$ ,  $\sigma_T(y)$  is a connected set.
- (iii)  $\sigma(T)$  is a connected set.

*Proof.* Suppose that  $\mathcal{H} = \bigvee \{k(\lambda) : \lambda \in B_a(T)\}$ , and  $Ty = \lambda y$  for some  $\lambda \in \mathbb{C}$  and  $y \in \mathcal{H}$ . For every  $\mu \in B_a(T) \setminus \{\lambda\}$ , we have,

$$\begin{aligned} \lambda \widehat{y}(\mu) &= \langle Ty, k(\mu) \rangle \\ &= \langle y, T^*k(\mu) \rangle \\ &= \mu \widehat{y}(\mu). \end{aligned}$$

Hence, the analytic function  $\widehat{y}$  is identically zero on  $B_a(T)$ ; this means that  $\langle y, k(\lambda) \rangle = 0$  for every  $\lambda \in B_a(T)$ . Since  $\mathcal{H} = \bigvee \{k(\lambda) : \lambda \in B_a(T)\}$ , we have

$y = 0$ . Thus  $\sigma_p(T) = \emptyset$ .

Now, suppose that  $\mathcal{H} = \bigvee\{k(\lambda) : \lambda \in G\}$  for some connected component  $G$  of  $B_a(T)$ .

(i) Suppose that there is an element  $y \in \mathcal{H}$  such that  $G \cap \rho_T(y) \neq \emptyset$ . Then there is an analytic  $\mathcal{H}$ -valued function  $f$  such that

$$(T - \lambda)f(\lambda) = y \text{ for } \lambda \in V,$$

where  $V = G \cap \rho_T(y)$ . Hence, for every  $\lambda \in V$ , we have

$$\begin{aligned} \widehat{y}(\lambda) &= \langle y, k(\lambda) \rangle \\ &= \langle (T - \lambda)f(\lambda), k(\lambda) \rangle \\ &= \langle f(\lambda), (T - \lambda)^*k(\lambda) \rangle \\ &= 0. \end{aligned}$$

Therefore,  $\widehat{y} \equiv 0$  on  $G$ . Since  $\mathcal{H} = \bigvee\{k(\lambda) : \lambda \in G\}$ , it follows that  $y = 0$ . Thus the first statement (i) holds.

(ii) Suppose that there is a non-zero element  $y \in \mathcal{H}$  for which  $\sigma_T(y)$  is disconnected, then  $\sigma_T(y) = \sigma_1 \cup \sigma_2$ , where  $\sigma_1$  and  $\sigma_2$  are two non-empty disjoint compact subsets of  $\mathbb{C}$ . Since  $\sigma_p(T) = \emptyset$ , we note that  $T$  has the single-valued extension property. And so, using the local version of Riesz's functional calculus, one shows that there are two non-zero elements  $y_1$  and  $y_2$  of  $\mathcal{H}$  such that  $y = y_1 + y_2$  and  $\sigma_T(y_i) \subset \sigma_i$ ,  $i = 1, 2$  which contradicts (i).

(iii) By (i), we have  $\bigcap_{y \in \mathcal{H} \setminus \{0\}} \sigma_T(y) \neq \emptyset$ , and  $\sigma_T(y)$  is connected for every  $y \in \mathcal{H} \setminus \{0\}$ . Since  $T$  has the single-valued extension property, we have  $\sigma(T) = \bigcup_{y \in \mathcal{H} \setminus \{0\}} \sigma_T(y)$  (see Proposition 1.3.2 of [13]). The result follows.  $\square$

**Corollary 5.2.** *Suppose that  $T$  possesses Bishop's property ( $\beta$ ). If  $\mathcal{H} = \bigvee\{k(\lambda) : \lambda \in B_a(T)\}$ , then  $B_a(T) = \Gamma(T) \setminus \sigma_{ap}(T)$ .*

We shall also need the following result due to L. R. Williams [21]. Recall that  $T$  is said to be pure if  $T$  does not have a non-zero reducing subspace  $M$  for which  $T$  is normal when restricted to  $M$ .

**Theorem 5.3.** *If  $T$  is a pure cyclic hyponormal operator so that  $\sigma_{ap}(T)$  has zero planar Lebesgue measure, then*

$$\mathcal{H} = \bigvee\{k(\lambda) : \lambda \in B_a(T)\}.$$

**Example 5.4.** Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ , and let  $\mathcal{H} = L_a^2(\mathbb{D})$  be the Bergman space, consisting of those analytic functions on  $\mathbb{D}$  that are square integrable on  $\mathbb{D}$  with respect to area measure. It is a Hilbert space when equipped with the inner product given by the formula

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dm(z), \quad f, g \in L_a^2(\mathbb{D}),$$

where  $dm$  denotes planar area measure, normalized so that  $\mathbb{D}$  has total mass 1. The *Bergman operator*  $S$  for  $\mathbb{D}$  is the operator multiplication by  $z$  on  $L_a^2(\mathbb{D})$ ; i.e.,  $(Sf)(z) = zf(z)$  for  $f \in L_a^2(\mathbb{D})$  and  $z \in \mathbb{D}$ . It is a pure cyclic subnormal operator, with cyclic vector the constant function 1, with  $\sigma(S) = \text{cl}(\mathbb{D})$  and  $\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$  (see [8]). In particular,  $B_a(S) = \sigma(S) \setminus \sigma_{ap}(S) = \mathbb{D} \neq \emptyset$ . By Theorem 5.3, we have

$$L_a^2(\mathbb{D}) = \bigvee \{k(\lambda) : \lambda \in B_a(S)\}.$$

This fact can be proved without using Theorem 5.3 and the vectors  $k(\lambda)$  can be given explicitly. To do this, fix  $\lambda \in \mathbb{D}$  and consider the power series expansion

$$(5.7) \quad k(\lambda)(z) = \sum_{n=0}^{+\infty} a_n z^n.$$

Fixing a non-negative integer  $n$ , and taking the polynomial  $p(z) = z^n$ , we have

$$\lambda^n = \langle p(S)1, k(\lambda) \rangle = \langle z^n, k(\lambda) \rangle = \int_{\mathbb{D}} z^n \left[ \sum_{i=0}^{+\infty} \overline{a_i z^i} \right] dm(z) = \pi \frac{\overline{a_n}}{n+1}.$$

So, by (5.7),  $k(\lambda)$  should be given by the formula

$$(5.8) \quad k(\lambda)(z) = \frac{1}{\pi} \sum_{n=0}^{+\infty} (n+1) (\overline{\lambda} z)^n = \frac{1}{\pi} (1 - \overline{\lambda} z)^{-2}.$$

The above infinite sum is evaluated by letting  $\omega = \overline{\lambda} z$  and noting that

$$\sum_{n=0}^{+\infty} (n+1) \omega^n = \left[ \sum_{n=0}^{+\infty} \omega^{n+1} \right]' = \left[ \frac{\omega}{1-\omega} \right]' = \frac{1}{(1-\omega)^2}.$$

Now, let  $f \in L_a^2(\mathbb{D})$  with a power series expansion

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n, \quad z \in \mathbb{D}.$$

It is easy to verify that the monomials  $(z^n)_{n \geq 0}$  form an orthogonal basis for  $L_a^2(\mathbb{D})$ , hence the partial sums of the above series converge to  $f$  in  $L_a^2(\mathbb{D})$ . Thus, for every  $\lambda \in \mathbb{D}$ , we have

$$\begin{aligned} \widehat{f}(\lambda) &= \langle f, k(\lambda) \rangle \\ &= \lim_{n \rightarrow +\infty} \left\langle \sum_{i=0}^n a_i z^i, k(\lambda) \right\rangle \\ &= \lim_{n \rightarrow +\infty} \sum_{i=0}^n a_i \lambda^i \\ &= f(\lambda). \end{aligned}$$

So, if  $f$  is orthogonal to  $\bigvee \{k(\lambda) : \lambda \in B_a(S)\}$ , then  $f$  must be identically zero. Therefore,  $L_a^2(\mathbb{D}) = \bigvee \{k(\lambda) : \lambda \in B_a(S)\}$ .

The following example shows that  $\bigvee\{k(\lambda) : \lambda \in B_a(S)\}$  needs not always be equal to  $\mathcal{H}$ . It also shows that the purity of  $T$  is necessary condition in Theorem 5.3.

**Example 5.5.** Let  $\delta$  be the  $\sigma$ -finite Dirac measure on  $\mathbb{C}$  at 2; i.e., for every subset  $A$  of  $\mathbb{C}$ ,

$$\delta(A) = \begin{cases} 1 & \text{if } 2 \in A \\ 0 & \text{otherwise} \end{cases}$$

The normal operator  $N_\delta : L^2(\delta) \rightarrow L^2(\delta)$  defined by  $N_\delta f(z) = zf(z)$  for all  $f \in L^2(\delta)$ , is cyclic and satisfies

$$\sigma(N_\delta) = \sigma_{ap}(N_\delta) = \sigma_p(N_\delta) = \{2\}.$$

Now, let  $\mathcal{H} = L^2_a(\mathbb{D}) \oplus L^2(\delta)$  and let  $T = S \oplus N_\delta$ , where  $S : L^2_a(\mathbb{D}) \rightarrow L^2_a(\mathbb{D})$  is the Bergman operator for  $\mathbb{D}$ . By Proposition 1-viii of [12],  $T$  is cyclic subnormal operator with cyclic vector  $1 \oplus 1$ . On the other hand, we have,

$$\sigma(T) = \text{cl}(\mathbb{D}) \cup \{2\}, \sigma_{ap}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{2\} \text{ and } \sigma_p(T) = \{2\}.$$

In particular,

$$B_a(T) = \sigma(T) \setminus \sigma_{ap}(T) = \mathbb{D} \neq \emptyset.$$

Since  $\sigma_p(T) \neq \emptyset$ , it then follows from Proposition 5.1 that

$$\bigvee\{k(\lambda) : \lambda \in B_a(T)\} \subsetneq \mathcal{H}.$$

Note that all the examples considered in the present paper are given either by weighted shift operators or by subnormal operators. So, it would be interesting to give other examples. We begin by recalling that an operator  $R \in \mathcal{L}(\mathcal{H})$  is said to be *Fredholm* if  $\text{ran}R$  is closed, and  $\ker R$  and  $\ker R^*$  are finite dimensional. The essential spectrum  $\sigma_e(R)$  of  $R$  is the set of all  $\lambda \in \mathbb{C}$  such that  $R - \lambda$  is not Fredholm; in fact,  $\sigma_e(R)$  is exactly the spectrum  $\sigma(\pi(R))$  in the Calkin algebra  $\mathcal{L}(\mathcal{H})/C$  of  $\pi(R)$ , where  $\pi$  is the natural quotient map of  $\mathcal{L}(\mathcal{H})$  onto  $\mathcal{L}(\mathcal{H})/C$ ; here,  $C$  denotes the ideal of all compact operators in  $\mathcal{L}(\mathcal{H})$ . Note that if  $R \in \mathcal{L}(\mathcal{H})$  is a cyclic operator such that  $\sigma_p(R) = \emptyset$ , then

$$\sigma_{ap}(R) = \sigma_e(R) = \{\lambda \in \mathbb{C} : \text{ran}(R - \lambda) \text{ is not closed}\}.$$

**Example 5.6.** Let  $S$  be the unweighted shift on  $\mathcal{H}$  i.e.,

$$Se_n = e_{n+1} \text{ for every non-negative integer } n,$$

where  $(e_n)_{n \geq 0}$  is an orthonormal basis of  $\mathcal{H}$ . It is shown in [11] that the operator  $T := S^* + 2S$  is hyponormal, but  $T^2$  is not. Therefore,  $T$  is not subnormal operator since every power of a subnormal operator is subnormal. On the other hand, it is easy to see that  $T$  is a cyclic operator. Indeed, by induction, one can show that  $e_k = p_k(T)e_0$  for every  $k \geq 0$ , where

$$(5.9) \quad \begin{cases} p_0(z) = 1, p_1(z) = \frac{1}{2}z \\ p_{k+1}(z) = \frac{1}{2} \left[ zp_k(z) - p_{k-1}(z) \right], \forall k \geq 1. \end{cases}$$

Hence,  $e_0$  is a cyclic vector for the operator  $T$ . Moreover, the following properties hold.

- (i)  $T$  is a pure cyclic hyponormal operator.
- (ii)  $\sigma_{ap}(T) = \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 = 1\}$ .
- (iii)  $\sigma(T) = \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 \leq 1\}$ . Hence,  $T$  is not a weighted shift operator since its spectrum is not a disc.
- (iv)  $B_a(T) = \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 < 1\}$  and for every  $\lambda \in B_a(T)$ ,

$$k(\lambda) = \sum_{n=0}^{+\infty} \overline{p_n(\lambda)} e_n,$$

where the polynomials  $p_n$  are given by (5.9).

- (v)  $\sigma_T(y) = \sigma(T)$  for every non-zero element  $y \in \mathcal{H}$ .

- (i) Let  $M$  be a reducing subspace of  $T$ . Since

$$S = \frac{2T - T^*}{3},$$

then  $M$  is a reducing subspace of  $S$ . And so  $M = \{0\}$  or  $M = \mathcal{H}$ . Hence  $T$  is pure operator.

(ii) Since,  $\sigma(\pi(S)) = \sigma_e(S) = \mathbb{T}$ , where  $\mathbb{T}$  is the unit circle of  $\mathbb{C}$ , and  $\pi(S)$  is a normal element in the Calkin algebra, it follows from the Spectral Mapping Theorem that

$$\begin{aligned} \sigma(\pi(T)) &= \{\bar{\lambda} + 2\lambda : \lambda \in \mathbb{T}\} \\ &= \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 = 1\}. \end{aligned}$$

Since  $\sigma_{ap}(T) = \sigma_e(T) = \sigma(\pi(T))$ , the desired result holds.

(iii) Note that the operator  $T$  is a Toeplitz operator  $T_\phi$  with associated function  $\phi = \bar{z} + 2z$ . It then follows from Theorem 5 of [19] that

$$\{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 > 1\} \subset \rho(T),$$

and either

$$\{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 < 1\} \subset \sigma(T) \text{ or } \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 < 1\} \subset \rho(T).$$

In view of Theorem 1 of [15], the last inclusion is impossible since  $T$  is not normal. Hence,  $\sigma(T) = \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 \leq 1\}$ .

- (iv) By Theorem 4.3, we have

$$B_a(T) = \sigma(T) \setminus \sigma_{ap}(T) = \{a + ib \in \mathbb{C} : (\frac{a}{3})^2 + b^2 < 1\}.$$

Now, let  $\lambda \in B_a(T)$ ; then we have

$$\begin{aligned} k(\lambda) &= \sum_{n=0}^{+\infty} \langle k(\lambda), e_n \rangle e_n \\ &= \sum_{n=0}^{+\infty} \langle k(\lambda), p_n(T)e_0 \rangle e_n \\ &= \sum_{n=0}^{+\infty} \overline{p_n(\lambda)} e_n. \end{aligned}$$

(v) By Theorem 5.3, we have

$$\mathcal{H} = \bigvee \{k(\lambda) : \lambda \in B_a(T)\}.$$

And so, by Proposition 5.1,  $\sigma(T) = \text{cl}(B_a(T)) \subset \sigma_{\mathcal{H}}(y)$  for every non-zero element  $y \in \mathcal{H}$ . The statement (v) is proved.

The following example shows that  $\mathcal{H} = \bigvee \{k(\lambda) : \lambda \in B_a(T)\}$  is not sufficient condition for  $T$  to have a connected spectrum, and gives a negative answer to Question B of [21] (see also [22]). The following lemma is needed.

**Lemma 5.7.** *Suppose that  $R \in \mathcal{L}(\mathcal{H})$ . If  $\sigma_R(y) = \sigma(R)$  for every non-zero element  $y \in \mathcal{H}$ , then  $\sigma(R)$  is connected.*

*Proof.* The proof is similar to the one of Proposition 5.1-(ii).  $\square$

**Example 5.8.** In considering the operator  $T$  given in Example 5.6, we let

$$\tilde{T} := (T - 4) \oplus (T + 4) \in \mathcal{L}(\tilde{\mathcal{H}}),$$

where  $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ . If we set

$$G^+ = \{a+ib \in \mathbb{C} : (\frac{a+4}{3})^2 + b^2 < 1\} \text{ and } G^- = \{a+ib \in \mathbb{C} : (\frac{a-4}{3})^2 + b^2 < 1\},$$

then the following hold.

- (i)  $\tilde{T}$  is a pure cyclic hyponormal operator with cyclic vector  $x = e_0 \oplus e_0$ .
- (ii)  $\sigma_{ap}(\tilde{T}) = \text{Fr}(G^-) \cup \text{Fr}(G^+)$ . Hence, by Theorem 5.3,

$$\tilde{\mathcal{H}} = \bigvee \{k_{\tilde{T}}(\lambda) : \lambda \in B_a(\tilde{T})\}.$$

- (iii)  $\sigma(\tilde{T}) = \text{cl}(G^-) \cup \text{cl}(G^+)$ . Hence,  $\sigma(\tilde{T})$  is disconnected; by Lemma 5.7, there is a non-zero element  $y \in \tilde{\mathcal{H}}$  for which  $\sigma_{\tilde{T}}(y) \subsetneq \sigma(\tilde{T})$ .
- (iv)  $B_a(\tilde{T}) = G^- \cup G^+$ .

What we need to show here is that  $x = e_0 \oplus e_0$  is a cyclic vector for  $\tilde{T}$ . Indeed, let

$$B^+ = \{\lambda \in \mathbb{C} : |\lambda + 4| < \frac{7}{2}\} \text{ and } B^- = \{\lambda \in \mathbb{C} : |\lambda - 4| < \frac{7}{2}\}.$$

Observe that  $B^- \cap B^+ = \emptyset$ , and consider the following analytic function on  $B := B^- \cup B^+$

$$f(z) = \begin{cases} 1 & \text{if } z \in B^+ \\ 0 & \text{if } z \in B^-. \end{cases}$$

Since  $\mathbb{C} \setminus B$  is connected, Runge's theorem shows that there is a sequence of polynomials  $(p_n)_n$  which converges uniformly to  $f$  on compact subsets of  $B$ . Hence,

$$p_n(\tilde{T})x \rightarrow f(\tilde{T})x = e_0 \oplus 0.$$

Since  $e_0$  is cyclic vector for  $T$ , for every  $y \in \mathcal{H}$ , we have

$$y \oplus 0 \in \bigvee \{p(\tilde{T})x : p \text{ polynomial}\}.$$

Similarly, for every  $y \in \mathcal{H}$ , we have

$$0 \oplus y \in \bigvee \{p(\tilde{T})x : p \text{ polynomial}\}.$$

Therefore,

$$\tilde{\mathcal{H}} = \bigvee \{p(\tilde{T})x : p \text{ polynomial}\}.$$

So,  $x = e_0 \oplus e_0$  is cyclic vector for  $\tilde{T}$ .

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