

Selections and order-like relations

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ABSTRACT. Every selection f for the family $\mathcal{F}_2(X)$ of at most two-point subsets of a set X naturally defines an order-like relation \preceq_f on X by $x \preceq_f y$ if and only if $f(\{x, y\}) = x$. In the present paper we study the relationship between \preceq_f and the possible topologies \mathcal{T} on X which realize the continuity of f with respect to the Vietoris topology $\tau_{V(\mathcal{T})}$ on $\mathcal{F}_2(X)$ generated by \mathcal{T} . We also study a similar problem about selections for the Vietoris hyperspace of all non-empty closed subsets of a Hausdorff space (X, \mathcal{T}) .

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1. INTRODUCTION

Let (X, \mathcal{T}) be a T_1 -space, where \mathcal{T} is the topology of X , and let $\mathcal{F}(X, \mathcal{T})$ be the family of all non-empty closed subsets of (X, \mathcal{T}) . Also, for every $n \geq 1$, let

$$\mathcal{F}_n(X) = \{S \subset X : 0 < |S| \leq n\}.$$

Note that $\mathcal{F}_n(X) \subset \mathcal{F}(X, \mathcal{T})$ because (X, \mathcal{T}) is a T_1 -space. Hence, we may consider $\mathcal{F}(X, \mathcal{T})$ as an extension of X identifying X with $\mathcal{F}_1(X)$. From this point of view, a topology τ on $\mathcal{F}(X, \mathcal{T})$ is *admissible* [8] if its restriction on X coincides with \mathcal{T} .

Let $\mathcal{D} \subset \mathcal{F}(X, \mathcal{T})$, and let τ be an admissible topology on $\mathcal{F}(X, \mathcal{T})$. A map $f : \mathcal{D} \rightarrow X$ is a *selection* for \mathcal{D} if $f(S) \in S$ for every $S \in \mathcal{D}$. A map $f : \mathcal{D} \rightarrow X$ is a τ -*continuous* selection for \mathcal{D} if it is a selection which is continuous with respect to the relative topology on \mathcal{D} as a subspace of $(\mathcal{F}(X, \mathcal{T}), \tau)$.

So far one of the best admissible topologies on $\mathcal{F}(X, \mathcal{T})$ is the Vietoris one $\tau_{V(\mathcal{T})}$. Let us recall that $\tau_{V(\mathcal{T})}$ is generated by all collections of the form

$$\langle \mathcal{V} \rangle = \left\{ S \in \mathcal{F}(X, \mathcal{T}) : S \cap V \neq \emptyset, V \in \mathcal{V}, \text{ and } S \subset \bigcup \mathcal{V} \right\},$$

where \mathcal{V} runs over the finite families of open subsets of (X, \mathcal{T}) .

Finally, let $Sel(X, \mathcal{T})$ be the set of all $\tau_{V(\mathcal{T})}$ -continuous selections for $\mathcal{F}(X, \mathcal{T})$, and $Sel_2(X, \mathcal{T})$ that of all $\tau_{V(\mathcal{T})}$ -continuous selections for $\mathcal{F}_2(X)$. Also, we will use $Sel_2(X)$ to denote the set of all selections for $\mathcal{F}_2(X)$. Note that always

$$Sel_2(X, \mathcal{T}) \subset Sel_2(X),$$

while

$$Sel_2(X, \mathcal{T}) \neq \emptyset \text{ provided } Sel(X, \mathcal{T}) \neq \emptyset.$$

In the present paper, we study relations between the set $Sel(X, \mathcal{T})$ (respectively, $Sel_2(X, \mathcal{T})$) and topological properties of (X, \mathcal{T}) . Suppose that (X, \mathcal{T}) is a space with $Sel(X, \mathcal{T}) \neq \emptyset$. The cardinality of the set $Sel(X, \mathcal{T})$ provides some information about (X, \mathcal{T}) but mainly when it is finite. For instance, if (X, \mathcal{T}) is connected, then $|Sel(X, \mathcal{T})| \leq 2$, [8], and if, in addition, (X, \mathcal{T}) is infinite, then it is compact if and only if $|Sel(X, \mathcal{T})| = 2$, [9]. On the other hand, $Sel(X, \mathcal{T})$ is finite if and only if (X, \mathcal{T}) has finitely many connected components, [10] (see, also, [3]).

In case $Sel(X, \mathcal{T})$ is infinite, it seems more reasonable to study the possible variety of elements of $Sel(X, \mathcal{T})$. The idea has led to several interesting characterizations of topological properties of (X, \mathcal{T}) based on different “extreme elements” of $Sel(X, \mathcal{T})$, see [3, 4, 5, 6]. In fact, the “extreme selections” are the first possible elements of $Sel(X, \mathcal{T})$ we may recognize. Namely, looking at our source, we may regard only a few possible ways to construct selections for $\mathcal{F}(X, \mathcal{T})$, and always the resulting selections have some extreme properties. That is, our knowledge about the variety of $Sel(X, \mathcal{T})$ is naturally associated to our constructions.

In this paper, we are concerned with somewhat different and somehow dual question following precisely the same idea. Suppose that (X, \mathcal{T}) is a space and $f \in Sel(X, \mathcal{T})$. Now, we become interested in the possible topologies $\tilde{\mathcal{T}}$ on X which preserve the continuity of f . The relation with the above classification problem is quite transparent. Namely, we may try to extract an information for a space (X, \mathcal{T}) that has a particular selection $f \in Sel(X, \mathcal{T})$, while the genesis of f could be related to another topology on X . Hence, \mathcal{T} may fail to be the proper topology on X generating the choice described by f . The best example is the set $Sel_2(X)$. In this case, any selection $f \in Sel_2(X)$ defines a natural topology \mathcal{T}_f on X (see Section 2), and once $f \in Sel_2(X, \mathcal{T})$ for some Hausdorff topology $\tilde{\mathcal{T}}$ on X , then $\mathcal{T}_f \subset \tilde{\mathcal{T}}$ (Theorem 3.5) and $f \in Sel_2(X, \tilde{\mathcal{T}})$ for any topology $\tilde{\mathcal{T}}$ on X which is finer than \mathcal{T} (Corollary 3.2). The case of selections for $\mathcal{F}(X, \mathcal{T})$ is discussed in the paper mainly for special extreme elements of $Sel(X, \mathcal{T})$ called *monotone selections*, see Section 5. For instance, we get that a space (X, \mathcal{T}) has a monotone selection $f \in Sel(X, \mathcal{T})$ if and only if (X, \mathcal{T}_f) is a *topologically well-orderable space* and \mathcal{T} is a *Sorgenfrey modification* of \mathcal{T}_f (Theorem 5.1). The paper contains also several examples demonstrating the importance of the hypotheses in our statements.

2. A TOPOLOGY GENERATED BY SELECTIONS

Every selection $f \in Sel_2(X)$ generates an order-like relation \preceq_f on X (see Michael [8]) defined for $x, y \in X$ by

$$x \preceq_f y \quad \text{if and only if} \quad f(\{x, y\}) = x.$$

In the sequel, we shall regard \preceq_f as an f -order on X . Also, let us agree to write that $x \prec_f y$ provided $x \preceq_f y$ and $x \neq y$.

Whenever $x \in X$, we consider the following f -intervals generated by a selection $f \in Sel_2(X)$:

$$\begin{aligned} (-\infty, x]_{\prec_f} &= \{z \in X : z \preceq_f x\} & \text{and} & & [x, +\infty)_{\prec_f} &= \{z \in X : x \preceq_f z\}, \\ (-\infty, x)_{\prec_f} &= \{z \in X : z \prec_f x\} & \text{and} & & (x, +\infty)_{\prec_f} &= \{z \in X : x \prec_f z\}. \end{aligned}$$

Proposition 2.1. *Let X be a T_1 -space, and let $f \in Sel_2(X)$. Then, whenever $x \in X$, the following holds.*

- (a) $(-\infty, x)_{\prec_f} \cap (x, +\infty)_{\prec_f} = \emptyset.$
- (b) $(-\infty, x]_{\prec_f} \cup [x, +\infty)_{\prec_f} = X.$

While the proof of Proposition 2.1 is trivial, we should be very careful working with f -intervals because, in general, \prec_f is not a linear order. For instance, if $x \prec_f y$, then it is not true that always

$$(-\infty, y]_{\prec_f} \cup [x, +\infty)_{\prec_f} = X$$

since $x \in (-\infty, y]_{\prec_f}$ does not imply that $(-\infty, x]_{\prec_f} \subset (-\infty, y]_{\prec_f}$.

Once we have associated some subsets of X generated by a selection $f \in Sel_2(X)$, we may associate a corresponding topology \mathcal{T}_f on X . Namely, let \mathcal{T}_f denote the topology on X generated by all f -intervals $(-\infty, x)_{\prec_f}$ and $(x, +\infty)_{\prec_f}$, $x \in X$.

Proposition 2.2. *Let X be a set, $f \in Sel_2(X)$, and let $x, y, z \in X$ be such that $z \prec_f x \prec_f y \prec_f z$. Then, $(-\infty, x]_{\prec_f} \cap [y, +\infty)_{\prec_f}$ is a clopen set of (X, \mathcal{T}_f) which separates z from the two-point set $\{x, y\}$.*

Proof. Note that $y \notin (-\infty, x]_{\prec_f}$ while $x \notin [y, +\infty)_{\prec_f}$. Hence,

$$(-\infty, x]_{\prec_f} \cap [y, +\infty)_{\prec_f} = (-\infty, x)_{\prec_f} \cap (y, +\infty)_{\prec_f},$$

which completes the proof. □

Here is another important property of \mathcal{T}_f .

Lemma 2.3. *Let X be a set, and let $f \in Sel_2(X)$. Then, \mathcal{T}_f is a Hausdorff topology on X .*

Proof. Take two different points $x, y \in X$, say $x \prec_f y$, and let us find disjoint sets $U, V \in \mathcal{T}_f$ such that $x \in U$ and $y \in V$. If

$$(x, y)_{\prec_f} = \{z \in X : x \prec_f z \prec_f y\} = \emptyset,$$

then

$$(-\infty, y)_{\prec_f} \cap (x, +\infty)_{\prec_f} = (x, y)_{\prec_f} = \emptyset,$$

so, in this case, we may merely take

$$U = (-\infty, y)_{\prec_f} \text{ and } V = (x, +\infty)_{\prec_f}.$$

If there exists a point $z \in (x, y)_{\prec_f}$, then, by Proposition 2.1, we may set

$$U = (-\infty, z)_{\prec_f} \text{ and } V = (z, +\infty)_{\prec_f}.$$

□

In what follows, to any selection $f \in \text{Sel}_2(X)$ we shall associate another selection $f^* : \mathcal{F}_2(X) \rightarrow X$ defined by $S = \{f(S), f^*(S)\}$ for any $S \in \mathcal{F}_2(X)$. Then, for $x, y \in X$, we have

$$x \prec_f y \text{ if and only if } y \prec_{f^*} x.$$

Hence, f^* generates the same family of f -intervals. That is, we have also the following result about \mathcal{T}_f .

Proposition 2.4. *Let X be a set, and let $f \in \text{Sel}_2(X)$. Then, $\mathcal{T}_f = \mathcal{T}_{f^*}$.*

3. WHICH TOPOLOGIES \mathcal{T} GIVE RISE TO $\tau_{V(\mathcal{T})}$ -CONTINUOUS SELECTIONS FOR $\mathcal{F}_2(X)$?

In this section we classify all possible Hausdorff topologies \mathcal{T} on a set (X, \mathcal{T}) with respect to which a given selection $f \in \text{Sel}_2(X)$ is $\tau_{V(\mathcal{T})}$ -continuous. The following theorem summarizes a well-known criterion of continuity of selections.

Theorem 3.1. *Let \mathcal{T} be a Hausdorff topology on a set X , and let $f \in \text{Sel}_2(X)$. The following two conditions are equivalent:*

- (a) $f \in \text{Sel}_2(X, \mathcal{T})$.
- (b) For every two points $x_1, x_2 \in X$, with $x_1 \prec_f x_2$, there are $V_1, V_2 \in \mathcal{T}$ such that $x_i \in V_i$, $i = 1, 2$, and $z_1 \prec_f z_2$ for every $z_i \in V_i$, $i = 1, 2$.

Proof. (a) \Rightarrow (b): Take points $x_1, x_2 \in X$, with $x_1 \prec_f x_2$. Since $x_1 \neq x_2$, by hypothesis, there are disjoint sets $U_1, U_2 \in \mathcal{T}$ such that $x_i \in U_i$, $i = 1, 2$. Since f is $\tau_{V(\mathcal{T})}$ -continuous and $f(\{x_1, x_2\}) = x_1 \in U_1$, this implies the existence of $V_1, V_2 \in \mathcal{T}$ such that $x_i \in V_i \subset U_i$, $i = 1, 2$, and $f(\langle V_1, V_2 \rangle) \subset U_1$. These V_1 and V_2 are as required in (b).

(b) \Rightarrow (a): First of all, note that f is $\tau_{V(\mathcal{T})}$ -continuous at the singletons of X . So, take two different point $x_1, x_2 \in X$, say $x_1 \prec_f x_2$. Then, by (b), there are $V_1, V_2 \in \mathcal{T}$ such that $x_i \in V_i$ and $z_1 \prec_f z_2$ for every $z_i \in V_i$, $i = 1, 2$. Hence, for every \mathcal{T} -neighbourhood U of x_1 , we have $f(\langle V_1 \cap U, V_2 \rangle) \subset U$. That is, f is $\tau_{V(\mathcal{T})}$ -continuous at $\{x_1, x_2\}$. □

Theorem 3.1 gives the following immediate consequence.

Corollary 3.2. *Let (X, \mathcal{T}) be a Hausdorff space, and let $f \in \text{Sel}_2(X, \mathcal{T})$. Then, $f \in \text{Sel}_2(X, \tilde{\mathcal{T}})$ for every topology $\tilde{\mathcal{T}}$ on X which is finer than \mathcal{T} .*

Corollary 3.2 suggests a natural question about a possible minimal topology \mathcal{T} on a set X such that a given selection $f \in \text{Sel}_2(X)$ is $\tau_{V(\mathcal{T})}$ -continuous. Namely,

Question 1. Let X be a set and $f \in \text{Sel}_2(X)$. Does there exist a topology \mathcal{T} on X which is minimal with respect to property “ $f \in \text{Sel}_2(X, \mathcal{T})$ ”?

Question 1 is related to the topology \mathcal{T}_f which is the only topological structure on X we may start with. Concerning \mathcal{T}_f , we have the following observations.

Lemma 3.3. Let (X, \mathcal{T}) be a T_1 -space and $f \in \text{Sel}_2(X, \mathcal{T})$. Then $(-\infty, x)_{\prec_f} \in \mathcal{T}$, whenever $x \in X$.

Proof. Take a point $x \in X$ and $z \in (-\infty, x)_{\prec_f}$. Since (X, \mathcal{T}) is a T_1 -space and $z \neq x$, there exists $U \in \mathcal{T}$ such that $z \in U$ and $x \notin U$. Since $f(\{z, x\}) = z$, there also exist $V, W \in \mathcal{T}$ such that $x \in V$, $z \in W$ and $f(\langle V, W \rangle) \subset U$. Then, $W \subset (-\infty, x)_{\prec_f}$. Indeed, take a point $y \in W$. Then, $\{x, y\} \in \langle V, W \rangle$, and therefore $f(\{x, y\}) \in U$. Hence, $f(\{x, y\}) = y$ which finally implies that $y \in (-\infty, x)_{\prec_f}$. \square

It should be mentioned that, in general, the f -intervals $(x, +\infty)_{\prec_f}$ may fail to be \mathcal{T} -open.

Example 3.4. There exists an infinite compact T_1 -space (X, \mathcal{T}) and $f \in \text{Sel}(X, \mathcal{T})$ such that $(x, +\infty)_{\prec_f} \notin \mathcal{T}$ for infinitely many points $x \in X$.

Proof. We take for X the set $\omega + 1$ with the cofinite topology \mathcal{T} , i.e.

$$\mathcal{T} = \{V \subset X : \text{either } V = \emptyset \text{ or } X \setminus V \text{ is finite}\}.$$

As it is well-known, the resulting space (X, \mathcal{T}) is a compact T_1 -space which is not Hausdorff. Let “ $<$ ” be the usual order on X . Define a selection f for $\mathcal{F}(X, \mathcal{T})$ by letting $f(F) = \max_{<} F$, $F \in \mathcal{F}(X, \mathcal{T})$. Let us show that f is $\tau_{V(\mathcal{T})}$ -continuous. Clearly, f is continuous at the singletons of X . Take a non-singleton $F \in \mathcal{F}(X, \mathcal{T})$ and a \mathcal{T} -neighbourhood V of $f(F)$. We distinguish the following two cases. If $F = X$, then $f(F) = \omega$, so there exists a point $x_F < \omega$ such that $\{x \in X : x_F < x\} \subset V$. If $F \neq X$, then F should be finite, so there exists $x_F = f(F \setminus f(F))$. Now, set $W = \{x \in V : x_F < x\}$, and then note that W remains a neighbourhood of $f(F)$. Let $\mathcal{U} = \{\{x\} \cup W : x \in F \text{ and } x \leq x_F\} \cup \{W\}$. Thus, we get a finite cover $\mathcal{U} \subset \mathcal{T}$ of F , with $F \cap U \neq \emptyset$ for every $U \in \mathcal{U}$. Hence $\langle \mathcal{U} \rangle$ is a $\tau_{V(\mathcal{T})}$ -neighbourhood of F in $\mathcal{F}(X, \mathcal{T})$. Take an $S \in \langle \mathcal{U} \rangle$. Then, $W \in \mathcal{U}$ implies $S \cap W \neq \emptyset$. According to the special choice of x_F , this implies that $f(S) = \max_{<} S \in W \subset V$. That is, f is $\tau_{V(\mathcal{T})}$ -continuous at F . To show the second part of our statement, take a point $x \in X$ such that $0 < x < \omega$. Now we have that

$$\begin{aligned} (x, +\infty)_{\prec_f} &= \{z \in X : x \prec_f z\} = \{z \leq \omega : z \neq x \text{ and } \max_{<} \{x, z\} = x\} \\ &= \{z \leq \omega : z < x\}. \end{aligned}$$

Hence, $(x, +\infty)_{\prec_f}$ fails to be \mathcal{T} -open as a non-empty finite set. \square

We are now ready for the following observation which sheds some light on Question 1.

Theorem 3.5. *For a T_1 -space (X, \mathcal{T}) and $f \in \text{Sel}_2(X, \mathcal{T})$, the following conditions are equivalent:*

- (a) (X, \mathcal{T}) is a Hausdorff space.
- (b) $f^* \in \text{Sel}_2(X, \mathcal{T})$.
- (c) $\mathcal{T}_f \subset \mathcal{T}$.

Proof. (a) \Rightarrow (b): Let (X, \mathcal{T}) be Hausdorff. To show that f^* is $\tau_{V(\mathcal{T})}$ -continuous, take points $x_1, x_2 \in X$ such that $x_1 \prec_f x_2$. Then, by Theorem 3.1, there are $V_1, V_2 \in \mathcal{T}$ such that $x_i \in V_i$, and $z_1 \prec_f z_2$ for every $z_i \in V_i$, $i = 1, 2$. Hence, $z_2 \prec_{f^*} z_1$ for every $z_i \in V_i$, $i = 1, 2$, and, by Theorem 3.1, f^* is $\tau_{V(\mathcal{T})}$ -continuous as well.

Since (b) \Rightarrow (c) follows by Lemma 3.3, while (c) \Rightarrow (a) by Lemma 2.3, the proof completes. \square

In view of Theorem 3.5 and Question 1, we may become inspired to ask if any selection $f \in \text{Sel}_2(X)$ is $\tau_{V(\mathcal{T}_f)}$ -continuous. The answer to this question is in negative.

Example 3.6. There exists a set X and $\sigma \in \text{Sel}_2(X)$ such that σ is not $\tau_{V(\mathcal{T}_\sigma)}$ -continuous.

Proof. Let $X = \{0, 1\} \times \omega$, and let “ $<$ ” be the usual order on ω . Define a selection $\sigma : \mathcal{F}_2(X) \rightarrow X$ as follows:

- (i) $\sigma(\{(0, n), (1, m)\}) = (1, m)$ if and only if $m = n + 1$ and $n \geq 1$,
- (ii) $\sigma(\{(0, n), (0, m)\}) = (0, m)$ for $0 < n \leq m$,
- (iii) $\sigma(\{(1, n), (1, m)\}) = (1, n)$ for $0 < n \leq m$,
- (iv) $\sigma(\{(0, 0), (0, m)\}) = (0, 0)$ for $m < \omega$,

and

- (v) $\sigma(\{(1, 0), (1, m)\}) = (1, m)$ for $m < \omega$.

Note that $(0, 0)$ is the \prec_σ -minimal point of X , while $(1, 0)$ is the \prec_σ -maximal one. Hence, all possible σ -intervals which may contain $(0, 0)$ are of the form

$$(-\infty, (0, n))_{\prec_\sigma} = \{(0, 0)\} \cup \{(0, m) : m \geq n + 1\} \cup \{(1, n + 1)\}, \quad n > 0,$$

and

$$(-\infty, (1, n + 1))_{\prec_\sigma} = \{(0, m) : m \neq n\} \cup \{(1, m) : m \leq n\}, \quad n \geq 0.$$

Since

$$(3.1) \quad \begin{aligned} (-\infty, (0, n))_{\prec_\sigma} \cap (-\infty, (0, n + 1))_{\prec_\sigma} &= \{(0, 0)\} \cup \{(0, m) : m \geq n + 2\} \\ (3.2) \quad &\subset (-\infty, (1, n + 1))_{\prec_\sigma}, \end{aligned}$$

the family $\{\{(0, 0)\} \cup \{(0, m) : m \geq k\} : k < \omega\}$ forms a local base at $(0, 0)$ in the topology \mathcal{T}_σ .

The situation with the point $(1, 0)$ is similar. Namely, all possible σ -intervals which may contain this point are of the form

$$((0, n), +\infty)_{\prec_\sigma} = \{(0, m) : 0 < m < n\} \cup \{(1, m) : m \neq n + 1\}, \quad n > 0,$$

and

$$((1, n + 1), +\infty)_{\prec_\sigma} = \{(0, n)\} \cup \{(1, m) : m \geq n + 2\} \cup \{(1, 0)\}, \quad n \geq 0.$$

Then, for every $n > 0$, we have

$$\begin{aligned} ((1, n), +\infty)_{\prec_\sigma} \cap ((1, n + 1), +\infty)_{\prec_\sigma} &= \{(1, m) : m \geq n + 2\} \cup \{(1, 0)\} \\ &\subset ((0, n), +\infty)_{\prec_\sigma}, \end{aligned}$$

so $\{\{(1, 0)\} \cup \{(1, m) : m \geq k\} : k < \omega\}$ is the corresponding local base at $(1, 0)$ in the topology \mathcal{T}_σ .

Now, we show that σ is not $\tau_{V(\mathcal{T}_\sigma)}$ -continuous. Recall that

$$\sigma(\{(0, 0), (1, 0)\}) = (0, 0).$$

Then, let

$$W = \{(0, 0)\} \cup \{(0, n) : n \geq 2\},$$

and let us show that $\sigma(\langle U, V \rangle) \not\subset W$ for any basic $\tau_{V(\mathcal{T}_\sigma)}$ -neighbourhood $\langle U, V \rangle$ of $\{(0, 0), (1, 0)\}$. Towards this end, for given neighbourhoods U of $(0, 0)$ and V of $(1, 0)$, pick a fixed $k < \omega$ such that

$$\{(0, n) : n \geq k\} \subset U \quad \text{and} \quad \{(1, n) : n \geq k\} \subset V.$$

Then, $\{(0, k), (1, k + 1)\} \in \langle U, V \rangle$, while

$$\sigma(\{(0, k), (1, k + 1)\}) = (1, k + 1) \notin W.$$

□

4. SELECTIONS GENERATED BY ORDER-LIKE RELATIONS

In what follows, all spaces are assumed to be at least Hausdorff. Below we summarize some known results about selections and special spaces.

Proposition 4.1. *Let (X, \mathcal{T}) be a compact space, and let $f \in \text{Sel}_2(X, \mathcal{T})$. Then, $\mathcal{T}_f = \mathcal{T}$ and, in particular, $f \in \text{Sel}_2(X, \mathcal{T}_f)$.*

Proof. By Lemma 2.3 and Theorem 3.5, \mathcal{T}_f is a Hausdorff topology with $\mathcal{T}_f \subset \mathcal{T}$, hence $\mathcal{T}_f = \mathcal{T}$. □

A similar result holds for connected spaces.

Proposition 4.2 ([8]). *Let (X, \mathcal{T}) be a connected space, and let $f \in \text{Sel}(X, \mathcal{T})$. Then, \prec_f is a proper linear order on X . In particular, (X, \mathcal{T}_f) is connected and $f \in \text{Sel}_2(X, \mathcal{T}_f)$.*

Finally, let us mention also the following simple observation which suggests a possible relationship between Propositions 4.1 and 4.2.

Proposition 4.3. *Let (X, \mathcal{T}) and $f \in \text{Sel}_2(X, \mathcal{T})$ be such that (X, \mathcal{T}_f) is connected. Then, for every $x, y \in X$, with $x \preceq_f y$, the interval*

$$[x, y]_{\prec_f} = \{z \in X : x \preceq_f z \preceq_f y\}$$

is a compact subset of (X, \mathcal{T}_f) .

Proof. By Proposition 2.2, \prec_f is a proper linear order on X . Take a \mathcal{T}_f -closed subset F of $[x, y]_{\prec_f}$. Then, $x \preceq_f z$ for every $z \in F$. Hence, with respect \prec_f , F is bounded below. Since $[x, y]_{\prec_f}$ is a connected subset of (X, \mathcal{T}_f) , this implies that F has a \prec_f -minimal element. In the same way, it has a \prec_f -maximal element. Hence, $[x, y]_{\prec_f}$ is compact in (X, \mathcal{T}_f) as a linear ordered space, see [2, 7]. \square

On the other hand, for compact spaces (X, \mathcal{T}) , there is a nice equilibrium between selections for $\mathcal{F}_2(X)$ and $\mathcal{F}(X, \mathcal{T})$.

Theorem 4.4 ([12]). *Let (X, \mathcal{T}) be a compact space. Then, $\text{Sel}_2(X, \mathcal{T}) \neq \emptyset$ if and only if $\text{Sel}(X, \mathcal{T}) \neq \emptyset$.*

In the connected case such a statement doesn't hold. For instance, the set of the real numbers \mathbb{R} endowed with the usual Euclidean topology \mathcal{E} has no $\tau_{V(\mathcal{E})}$ -continuous selection for $\mathcal{F}(\mathbb{R}, \mathcal{E})$, [1]. Here is a simple characterization for the genesis of selections for the closed subsets of connected spaces. In what follows, we shall say that $x \in X$ is a \prec_f -extreme point of X , where $f \in \text{Sel}_2(X)$, if either $x \preceq_f z$ for every $z \in X$ or $z \preceq_f x$ for every $z \in X$.

Theorem 4.5. *Let (X, \mathcal{T}) and $f \in \text{Sel}_2(X, \mathcal{T})$ be such that (X, \mathcal{T}_f) is connected. Then, $\text{Sel}(X, \mathcal{T}_f) \neq \emptyset$ if and only if (X, \mathcal{T}_f) has at least one \prec_f -extreme point.*

Proof. Let $h \in \text{Sel}(X, \mathcal{T}_f)$. According to a result of Michael [8], either

$$h|_{\mathcal{F}_2(X)} = f \text{ or } h|_{\mathcal{F}_2(X)} = f^*.$$

On the other hand, $h([x, +\infty)_{\prec_h}) = x$ for every $x \in X$. Hence, $h(X) = \min_{\prec_h} X$, and therefore $h(X)$ is a \prec_f -extreme point for X as well.

Suppose now that X has a \prec_f -extreme point, say there exists $\min_{\prec_f} X$. Then, by Proposition 4.3, for every $F \in \mathcal{F}(X, \mathcal{T}_f)$ there exists $\min_{\prec_f} F$. Hence, we may define a selection h for $\mathcal{F}(X, \mathcal{T}_f)$ by

$$h(F) = \min_{\prec_f} F, \quad F \in \mathcal{F}(X, \mathcal{T}_f).$$

The verification that $h \in \text{Sel}(X, \mathcal{T}_f)$ is well-known. \square

In view of Theorem 4.5, a possible equilibrium between the compact and connected cases is given by the existence of selections with an additional compact-type of property. Towards this end, let us recall that the *Fell topology* $\tau_{F(\mathcal{T})}$ on $\mathcal{F}(X, \mathcal{T})$ is defined by all basic $\tau_{V(\mathcal{T})}$ -neighbourhood $\langle \mathcal{V} \rangle$ such that $X \setminus \bigcup \mathcal{V}$ is compact in (X, \mathcal{T}) .

Corollary 4.6. *Let (X, \mathcal{T}) and $f \in \text{Sel}_2(X, \mathcal{T})$ be such that (X, \mathcal{T}_f) is connected. Then, $\text{Sel}(X, \mathcal{T}_f) \neq \emptyset$ if and only if $\mathcal{F}_2(X)$ has a $\tau_{\mathcal{F}(\mathcal{T}_f)}$ -continuous selection.*

Proof. If $h \in \text{Sel}(X, \mathcal{T}_f)$, then there exists $\min_{\prec_g} X$, where $g = h|_{\mathcal{F}_2(X)}$. Hence, g is a $\tau_{\mathcal{F}(\mathcal{T}_f)}$ -continuous selection for $\mathcal{F}_2(X)$ because, by Proposition 4.3, $X \setminus (x, +\infty)_{\prec_g}$ is compact in (X, \mathcal{T}_f) for every $x \in X$. If g is a $\tau_{\mathcal{F}(\mathcal{T}_f)}$ -continuous selection for $\mathcal{F}_2(X)$, then, by [4, Lemma 4.1], there exists $\min_{\prec_g} X$. Hence, by Theorem 4.5, $\text{Sel}(X, \mathcal{T}_f) \neq \emptyset$. \square

In view of Theorems 4.4 and 4.5, the following question becomes interesting.

Question 2. Let (X, \mathcal{T}) be a connected space and $f \in \text{Sel}_2(X, \mathcal{T})$. Is it true that $\text{Sel}(X, \mathcal{T}) \neq \emptyset$ if and only if $\text{Sel}(X, \mathcal{T}_f) \neq \emptyset$?

In general, the answer is in negative which is the purpose of the next sections of the paper.

4.1. Monotone selections. Let (X, \mathcal{T}) be a space. We shall say that a selection f for $\mathcal{F}(X, \mathcal{T})$ is *monotone* if, for every $G, F \in \mathcal{F}(X, \mathcal{T})$, $f(F) \in G \subset F$ implies $f(G) = f(F)$.

First of all, let us observe that every monotone selection is Vietoris continuous.

Proposition 4.7. *Let (X, \mathcal{T}) be a space, and let f be a monotone selection for $\mathcal{F}(X, \mathcal{T})$. Then, f is $\tau_{V(\mathcal{T})}$ -continuous.*

Proof. Let \prec_f be the f -order generated by $f|_{\mathcal{F}(X, \mathcal{T})}$, and let us observe that $f(F) = \min_{\prec_f} F$ for every $F \in \mathcal{F}(X, \mathcal{T})$. Indeed, take an $F \in \mathcal{F}(X, \mathcal{T})$ and a point $x \in F$. Then, $G = \{x, f(F)\} \subset F$ and $f(F) \in G$, so, by definition, $f(G) = f(F)$. Thus, $f(F) \preceq_f x$, i.e. $f(F) = \min_{\prec_f} F$. Now, we show that f is continuous at F . To this end, we consider only the case of a non-singleton F . Take a neighbourhood $V \in \mathcal{T}$ of $f(F)$, with $F \setminus V \neq \emptyset$. Next, let

$$U_1 = \min_{\prec_f} (-\infty, y)_{\prec_f} \cap V \text{ and } U_2 = (f(F), +\infty)_{\prec_f},$$

where $y = \min_{\prec_f} [f(F), +\infty)_{\prec_f} \setminus V$. Then, $\langle U_1, U_2 \rangle$ is a $\tau_{V(\mathcal{T})}$ -neighbourhood of F , with $f(\langle U_1, U_2 \rangle) \subset V$, because $S \in \langle U_1, U_2 \rangle$ implies $f(S) = \min_{\prec_f} S \in U_1 \subset V$. \square

We proceed to some basic properties of monotone selections. To this end, we need a bit more terminology concerning special selections. We shall say that a selection f for $\mathcal{F}(X, \mathcal{T})$ is *weakly monotone* if $f(F \cup G) = f(F)$, whenever $G, F \in \mathcal{F}(X, \mathcal{T})$ and $f(F) = f(G)$. Also, let us agree that a selection f for $\mathcal{F}_2(X)$ is *transitively regular* provided the f -order \prec_f on X is transitive, i.e. for every $x, y, z \in X$, $f(\{x, y\}) = x$ and $f(\{y, z\}) = y$ imply $f(\{x, z\}) = x$. In case f is a selection for $\mathcal{F}(X, \mathcal{T})$ we shall use the same term “transitively regular” to suggest that $f|_{\mathcal{F}_2(X)}$ is transitively regular.

Proposition 4.8. *Let (X, \mathcal{T}) be a space, and let f be a monotone selection for $\mathcal{F}(X, \mathcal{T})$. Then, f is weakly monotone and transitively regular.*

Proof. Suppose that $G, F \in \mathcal{F}(X, \mathcal{T})$ and $f(F) = f(G)$. Then,

$$\text{either } f(F \cup G) \in F \subset F \cup G \text{ or } f(F \cup G) \in G \subset F \cup G.$$

Hence, $f(F \cup G) \in \{f(F), f(G)\}$, so f is weakly monotone.

Suppose now that $x, y, z \in X$ are such that $f(\{x, y\}) = x$ and $f(\{y, z\}) = y$. Then, $s = f(\{x, y, z\}) \in \{x, y, z\}$ implies that $f(\{s, t\}) = s$ for every $t \in \{x, y, z\}$. In this case, the only possibility is $s = x$. That is, f is transitively regular as well. \square

In what follows, let $\mathcal{F}_{<\omega}(X) = \bigcup\{\mathcal{F}_n(X) : 0 < n < \omega\}$.

Lemma 4.9. *Let (X, \mathcal{T}) be a space, and let $f \in \text{Sel}(X, \mathcal{T})$ be weakly monotone and transitively regular. Then, $f(S) = \min_{\prec_f} S$ for every $S \in \mathcal{F}(X, \mathcal{T})$.*

Proof. Take a $T \in \mathcal{F}_{<\omega}(X)$. Since f is transitively regular, there exists $\min_{\prec_f} T$. On the other hand, there exists a finite subset $\mathcal{L} \subset \mathcal{F}_2(X)$ such that $T = \bigcup \mathcal{L}$ and $\min_{\prec_f} L = \min_{\prec_f} T$ for every $L \in \mathcal{L}$. Then, $f(T) = f(\bigcup \mathcal{L}) = \min_{\prec_f} T$ because f is weakly monotone. Since $\mathcal{F}_{<\omega}(X)$ is $\tau_{V(\mathcal{T})}$ -dense in $\mathcal{F}(X, \mathcal{T})$ and f is $\tau_{V(\mathcal{T})}$ -continuous, this finally completes the proof. Indeed, suppose that, in the opposite, there exists an $S \in \mathcal{F}(X, \mathcal{T})$ and a point $x \in S$ such that $x \prec_f f(S)$. Then, by Theorem 3.1, there exists a \mathcal{T} -neighbourhood V of $f(S)$ such that $x \prec_f y$ whenever $y \in V$. Since f is $\tau_{V(\mathcal{T})}$ -continuous, there also exists a finite open cover $\mathcal{U} \subset \mathcal{T}$ of S such that $f(\langle \mathcal{U} \rangle) \subset V$. Take a finite subset $T \subset S$ such that $x \in T$ and $T \in \langle \mathcal{U} \rangle$. Then, from one hand, $f(T) \in V$ implies that $x \prec_f f(T)$. However, from another hand, $f(T) = \min_{\prec_f} T \preceq_f x$ because $x \in T$. A contradiction. \square

Corollary 4.10. *Let (X, \mathcal{T}) be a space, and let $f \in \text{Sel}(X, \mathcal{T})$. Then f is monotone if and only if it is weakly monotone and transitively regular.*

Proof. By Proposition 4.8, it suffices to show that f is monotone provided it is weakly monotone and transitively regular. So, take $F, G \in \mathcal{F}(X, \mathcal{T})$ such that $f(F) \in G \subset F$. Then, by Lemma 4.9, $f(F) = \min_{\preceq_f} F$. Since $f(F) \in G$ and $G \subset F$, this implies that $f(F) = \min_{\prec_f} G$. According to Lemma 4.9 once again, we get that $f(G) = f(F)$. That is, f is monotone. \square

The statements of Corollary 4.10 seem to be the best possible. Turning to this, for every point $x \in X$ we consider the *component* $\mathcal{C}[x]$ of this point in (X, \mathcal{T}) defined by

$$\mathcal{C}[x] = \bigcup\{C \subset X : x \in C \text{ and } C \text{ is connected}\}.$$

and, respectively, the *quasi component* $\mathcal{C}^*[x]$ defined by

$$\mathcal{C}^*[x] = \bigcap\{C \subset X : x \in C \text{ and } C \text{ is clopen}\}.$$

According to [6, Theorem 4.1], $\mathcal{C}^*[x] = \mathcal{C}[x]$ for every $x \in X$ provided

$$\text{Sel}_2(X, \mathcal{T}) \neq \emptyset.$$

Proposition 4.11. *Let (X, \mathcal{T}) be a space which has a selection $f \in \text{Sel}_2(X, \mathcal{T})$ that is not transitively regular. Then, $|\{C^*[x] : x \in X\}| \geq 3$.*

Proof. By condition, there are points $x, y, z \in X$ such that $z \prec_f y \prec_f x \prec_f z$. Therefore, we have also that $y \prec_f x \prec_f z \prec_f y$ and $x \prec_f z \prec_f y \prec_f x$. Hence, by Proposition 2.2 and Theorem 3.5,

$$V_z = (-\infty, y]_{\prec_f} \cap [x, +\infty)_{\prec_f}, \quad V_y = (-\infty, x]_{\prec_f} \cap [z, +\infty)_{\prec_f}$$

and

$$V_x = (-\infty, z]_{\prec_f} \cap [y, +\infty)_{\prec_f}$$

are \mathcal{T} -clopen pairwise disjoint neighbourhoods of z, y and, respectively, x . \square

On the base of Proposition 4.11 we now have the following two simple examples.

Example 4.12. There exists a compact metric space (X, \mathcal{T}) and a transitively regular selection $f \in \text{Sel}(X, \mathcal{T})$ which is not weakly monotone.

Proof. Let $X = \mathbb{I}_1 \oplus \mathbb{I}_2 \oplus \mathbb{I}_3$, where $\mathbb{I}_i = [0, 1]$ for every $i \leq 3$. We consider each \mathbb{I}_i endowed with the usual Euclidean topology \mathcal{T}_i , while the topology \mathcal{T} on X is just the disjoint sum of these topologies. Next, for every i , let $f_i \in \text{Sel}(\mathbb{I}_i, \mathcal{T}_i)$ be the standard selection $f_i(S) = \min S, S \in \mathcal{F}(\mathbb{I}_i, \mathcal{T}_i)$. Also, for every $S \in \mathcal{F}(X, \mathcal{T})$, let $i(S) = \min\{j \leq 3 : S \cap \mathbb{I}_j \neq \emptyset\}$. Then, define a selection $f \in \text{Sel}(X, \mathcal{T})$ by letting for every $S \in \mathcal{F}(X, \mathcal{T})$ that $f(S) = f_2(S \cap \mathbb{I}_2)$ provided $S \cap \mathbb{I}_i \neq \emptyset$ for every $i \leq 3$, and $f(S) = f_{i(S)}(S \cap \mathbb{I}_{i(S)})$ otherwise. This f is the required one. \square

Example 4.13. There exists a compact metric space (X, \mathcal{T}) and a weakly monotone selection $f \in \text{Sel}(X, \mathcal{T})$ which is not transitively regular.

Proof. Take merely $X = \{1, 2, 3\}$ with the usual discrete topology \mathcal{T} . Then, define the required selection $f \in \text{Sel}(X, \mathcal{T})$ by $f(\{1, 2\}) = 1, f(\{2, 3\}) = 2, f(\{1, 3\}) = 3$ and $f(\{1, 2, 3\}) = 3$. \square

5. WHICH TOPOLOGIES \mathcal{T} GIVE RISE TO $\tau_{V(\mathcal{T})}$ -CONTINUOUS SELECTIONS FOR $\mathcal{F}(X, \mathcal{T})$?

We are now ready to provide an answer to Question 2. Let X be a space, \prec be a linear order on X , and let \mathcal{T}_\prec be the topology on X generated by \prec . Let us recall that (X, \mathcal{T}_\prec) is a *topologically well-ordered space* (see [1]) if every non-empty closed subset of (X, \mathcal{T}_\prec) has a minimal element. Now, we shall say that a space (X, \mathcal{T}) is a *Sorgenfrey well-orderable* if there exists a linear order \prec on X such that

- (i) (X, \mathcal{T}_\prec) is topologically well-ordered, with $\mathcal{T}_\prec \subset \mathcal{T}$, and
- (ii) for every point $x \in X$, with $x \prec \sup_\prec X$, and its neighbourhood $V \in \mathcal{T}$ there exists a point $y \in (x, +\infty)_\prec$ such that $[x, y)_\prec \subset V$.

In this case, we shall say that the topology \mathcal{T} is a *Sorgenfrey modification* of \mathcal{T}_\prec .

Theorem 5.1. *For a Hausdorff space (X, \mathcal{T}) , the following conditions are equivalent:*

- (a) $\mathcal{F}(X, \mathcal{T})$ has a monotone selection.
- (b) (X, \mathcal{T}) is a Sorgenfrey well-orderable space.

Proof. (a) \Rightarrow (b): Let f be a monotone selection for $\mathcal{F}(X, \mathcal{T})$. Then, by Proposition 4.7, $f \in \text{Sel}(X, \mathcal{T})$, while, by Lemma 4.9 and Corollary 4.10, \prec_f is a linear order on X so that $f(S) = \min_{\prec_f} S$ for every $S \in \mathcal{F}(X, \mathcal{T})$. Hence, (X, \mathcal{T}_f) is a topologically well-ordered space and, by Theorem 3.5, $\mathcal{T}_f \subset \mathcal{T}$. Take $V \in \mathcal{T}$ and $x \in V$, with $x \prec_f \sup_{\prec_f} X$. If $[x, +\infty)_{\prec_f} \subset V$, take an arbitrary point $y \in (x, +\infty)_{\prec_f}$. Otherwise, let $y = f([x, +\infty)_{\prec_f} \setminus V)$. According to Lemma 4.9, we always have $[x, y)_{\prec_f} \subset V$, so \mathcal{T} is a Sorgenfrey modification of \mathcal{T}_f .

(b) \Rightarrow (a): Suppose now that (X, \mathcal{T}) is a Sorgenfrey well-ordered space, and let \prec be as in (i) and (ii). Take an $F \in \mathcal{F}(X, \mathcal{T})$, and let us show that F has a first element with respect to \prec . If $x_0 = \min_{\prec} X \in F$, then $\min_{\prec} F = x_0$. If $x_0 \notin F$, then

$$U = \bigcup \{ [x_0, y)_{\prec} : y \in X \text{ and } [x_0, y)_{\prec} \cap F = \emptyset \}$$

is a \mathcal{T}_{\prec} -open set in X such that $y \prec z$ for every $y \in U$ and $z \in F$. Since $F \subset X \setminus U$, by (i), there exists $x = \min_{\prec}(X \setminus U)$. This x is the required \prec -minimal element of F . Indeed, if $x \notin F$, then, by (ii), there should exist an $y \in (x, +\infty)_{\prec}$ such that $[x, y)_{\prec} \cap F = \emptyset$. However, this will imply that $x \in [x_0, y)_{\prec} \subset U$ which is impossible. Thus, $x = \min_{\prec} F$. Having already established this, we may define a selection f for $\mathcal{F}(X, \mathcal{T})$ by $f(F) = \min_{\prec} F$ for every $F \in \mathcal{F}(X, \mathcal{T})$. Finally, we show that f is monotone. Take $F, G \in \mathcal{F}(X, \mathcal{T})$ such that $f(F) \in G \subset F$. Then, just like in Corollary 4.10, $f(F) = \min_{\prec} F \preceq \min_{\prec} G \preceq f(G)$ because $f(F) \in G$ and $G \subset F$. \square

Example 5.2. There exists a hereditarily separable, Lindelöf, perfectly normal, strongly paracompact, strongly zero-dimensional and non-Čech complete space (Z, \mathcal{T}) such that $\mathcal{F}(Z, \mathcal{T})$ has a monotone selections.

Proof. For (Z, \mathcal{T}) we may take the usual *Sorgenfrey line* [11] (see, also, [2]). Note that the Sorgenfrey line is a Sorgenfrey modification of of the interval $[0, 1)$ endowed with the usual Euclidean topology \mathcal{T}_e . Since, $([0, 1), \mathcal{T}_e)$ is a topologically well-ordered space, by Theorem 5.1, $\mathcal{F}(Z, \mathcal{T})$ has a monotone selections. Also, the Sorgenfrey line has all other properties, see [2]. \square

Finally, we provide the promised negative answer to Question 2.

Example 5.3. There exists a separable, connected and metrizable space (X, \mathcal{T}) such that

- (i) $\text{Sel}_2(X, \mathcal{T}) \neq \emptyset$,
- (ii) $\text{Sel}(X, \mathcal{T}_f) \neq \emptyset$ for every $f \in \text{Sel}_2(X, \mathcal{T})$,
- (iii) $\text{Sel}(X, \mathcal{T}) = \emptyset$.

Proof. Consider the subset

$$X = \left\{ (t, s) \in \mathbb{R}^2 : (t, s) = (0, 0) \text{ or } t \in [-1, 0) \cup (0, 1] \text{ and } s = \sin \frac{1}{t} \right\}$$

of the Euclidean plane \mathbb{R}^2 with the usual topology \mathcal{T} as a subspace. Then, (X, \mathcal{T}) is separable, metrizable and connected. Note that the projection $\pi : X \rightarrow [-1, 1]$ onto the first factor is a continuous bijection, where $[-1, 1]$ is endowed with the Euclidean topology \mathcal{T}_e . Claim that $\text{Sel}_2(X, \mathcal{T}) \neq \emptyset$. Indeed, let $g \in \text{Sel}_2([-1, 1], \mathcal{T}_e)$, and let $\tilde{\mathcal{T}}_e = \pi^{-1}(\mathcal{T}_e)$. Clearly $\tilde{\mathcal{T}}_e$ is a topology on X which is coarser than \mathcal{T} . On the other hand, $h(S) = \pi^{-1} \circ g \circ \pi(S)$, $S \in \mathcal{F}_2(X)$, defines a selection for $\mathcal{F}_2(X)$ which is $\tau_{V(\tilde{\mathcal{T}}_e)}$ -continuous because $g \in \text{Sel}_2([-1, 1], \mathcal{T}_e)$. Therefore, by Corollary 3.2, h is a $\tau_{V(\mathcal{T})}$ -continuous selection for $\mathcal{F}_2(X)$. Thus, (i) holds. Next, let us show that $\tilde{\mathcal{T}}_e = \mathcal{T}_f$ for every $f \in \text{Sel}_2(X, \mathcal{T})$. Namely, if $f \in \text{Sel}_2(X, \mathcal{T})$, then $k(S) = \pi \circ f \circ \pi^{-1}(S)$, $S \in \mathcal{F}_2([-1, 1])$, defines a selection for $\mathcal{F}_2([-1, 1])$. Let $h \in \text{Sel}_2(X, \mathcal{T})$ be defined as above, i.e. by the fixed selection $g \in \text{Sel}_2([-1, 1], \mathcal{T}_e)$. Then, by a result of Michael [8], either $f = h$ or $f = h^*$ because (X, \mathcal{T}) is connected. Therefore, either $k = g$ or $k = g^*$, so, by Theorem 3.5, $k \in \text{Sel}_2([-1, 1], \mathcal{T}_e)$. Thus, by Corollary 4.1, we have $\tilde{\mathcal{T}}_e = \mathcal{T}_f$ for every $f \in \text{Sel}_2(X, \mathcal{T})$ because $\tilde{\mathcal{T}}_e$ is a compact topology on X . In particular, by Theorem 4.4, this provides (ii). Now, if $f \in \text{Sel}(X, \mathcal{T})$, then, by a result of [8], f should be monotone because (X, \mathcal{T}) is connected. Then, by Theorem 5.1, \mathcal{T} should a Sorgenfrey modification of $\mathcal{T}_f = \tilde{\mathcal{T}}_e$. However, this is impossible because there exists a \mathcal{T} -neighbourhood V of $\vartheta = (0, 0)$ such that $[\vartheta, x)_{\prec_f} \setminus V \neq \emptyset$ for every $x \in X$ with $\vartheta \prec_f x$. This contradiction demonstrates (iii) which completes the proof. \square

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