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Two general fixed point theorems for a sequence of mappings satisfying implicit relations in Gp - metric spaces

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ABSTRACT

In this paper, two general fixed point theorem for a sequence of mappings satisfying implicit relations in Gp - complete metric spaces are proved.

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1. INTRODUCTION AND PRELIMINARIES

In this paper we shall investigate the existence and uniqueness of common fixed point of mappings via implicit relations in the setting of Gp - metric spaces, inspired from the notion of Gp -metric spaces [25],[4],[6],[7] and other papers. We remind that Gp - metric is inspired from the notions of G - metric ([15],[16],[1],[3],[14] and other papers) and partial metric ([13], [1], [2], [8], [9], [10], [11], [12] and other papers).

Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [17], [18]. Some fixed point theorems for mappings satisfying a implicit relation in G - metric spaces are established in [19] - [22]. Recently, fixed point for mappings satisfying implicit relation in partial metric spaces are obtained in

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[5], [9], [10], [24]. Quite recently, a fixed point result for mappings satisfying an implicit relation in Gp - metric spaces is obtained in [23]. We first recall the notion of Gp - metric.

Definition 1.1 ([25]). Let X be a nonempty set. A function $Gp : X^3 \to \mathbb{R}_+$ is called a Gp - metric on X if the following conditions are satisfied:

(Gp_1) : $x = y = z$ if $Gp(x, y, z) = Gp(x, x, x) = Gp(y, y, y) = Gp(z, z, z)$,
(Gp_2) : $0 \leq Gp(x, x, x) \leq Gp(x, x, y) \leq Gp(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$,
(Gp_3) : $Gp(x, y, z) = Gp(y, z, x) = \ldots$ (symmetry in all three variables),
(Gp_4) : $Gp(x, y, z) \leq Gp(x, a, a)+Gp(a, y, z)-Gp(a, a, a)$ for all $x,\ y,\ z,\ a \in X$.

The pair (X, Gp) is called a Gp - metric space.

Definition 1.2 ([25]). Let (X, Gp) be a Gp - metric space and $\{x_n\}$ a sequence in X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ ($\{x_n\}$ is Gp - convergent to x) if $\lim_{n,m\to\infty} Gp(x, x_n, x_m) = Gp(x, x, x)$.

Theorem 1.3 ([4]). *Let (X, Gp) be a Gp - metric space. Then, for any $\{x_n\} \in X$ and $x \in X$, the following conditions are equivalent:*
a) $\{x_n\}$ is Gp - convergent to x,
b) $Gp(x_n, x_n, x) \to Gp(x, x, x)$ as $n \to \infty$,
c) $Gp(x_n, x, x) \to Gp(x, x, x)$ as $n \to \infty$.

Definition 1.4 ([25]). Let (X, Gp) be a Gp - metric space.

1) A sequence $\{x_n\}$ of X is called a Gp - Cauchy sequence in X if $\lim_{n,m\to\infty} GP(x_n, x_m, x_m)$ exists and is finite.

2) A Gp - metric space is said to be Gp - complete if every Gp - Cauchy sequence in X, Gp - converges to $x \in X$ such that $\lim_{n,m\to\infty} Gp\,(x_n, x_m, x_m) = Gp(x, x, x)$.

Lemma 1.5 ([4]). *Let (X, Gp) be a Gp - metric space. Then:*
1) If $Gp(x, y, z) = 0$ then $x = y = z$,
2) If $x \neq y$ then $Gp(x, x, y) > 0$.

Lemma 1.6. *Let (X, Gp) be a Gp - metric space and $\{x_n\}$ is a sequence in X which is G_p - convergent to a point $x \in X$ with $G_p\,(x, x, x) = 0$. Then, $\lim_{n\to\infty} G\,(x_n, y, z) = G\,(x, y, z)$ for all $y, z \in X$.*

Proof. By (Gp_4)

$$\begin{aligned} Gp\,(x, y, z) &\leq Gp\,(x, x_n, x_n) + Gp\,(x_n, y, z) - Gp\,(x_n, x_n, x_n) \\ &\leq Gp\,(x, x_n, x_n) + Gp\,(x_n, y, z)\,, \end{aligned} \tag{1.1}$$

which implies

$$\begin{aligned} Gp\,(x, y, z) - Gp\,(x, x_n, x_n) &\leq Gp\,(x_n, y, z) \\ &\leq Gp\,(x_n, x, x) + Gp\,(x, y, z)\,. \end{aligned}$$

By Theorem 1.3,

$$Gp\,(x_n, x, x) \to Gp\,(x, x, x) = 0$$

and

$$Gp(x, x_n, x_n) \to Gp(x, x, x) = 0.$$

Letting n tends to infinity in (1.1) we obtain

$$\lim_{n\to\infty} Gp(x_n, y, z) = Gp(x, y, z).$$

□

Quite recently, Meena and Nema [14] initiated the study of fixed points for sequences of mappings in G - metric spaces.

2. Implicit relations

Definition 2.1. Let $\mathfrak{F}_{Gp}$ be the set of all continuous functions $F(t_1, ..., t_5)$: $\mathbb{R}_+^5 \to \mathbb{R}$ satisfying the following conditions:

(F_1) : F is non - increasing in variables t_2, t_3, t_4, t_5,

(F_2) : There exists $h \in [0, 1)$ such that for all $u, v \geq 0$, $F(u, v, u, v, u+v) \leq 0$ implies $u \leq hv$.

In the following examples, the proofs of property (F_1) are obviously.

Example 2.2. $F(t_1, ..., t_5) = t_1 - at_2 - bt_3 - ct_4 - dt_5$, where $a, b, c, d \geq 0$ and $a + b + c + 2d < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, u+v) = u - av - bu - cv - d(u+v) \leq 0$, which implies $u \leq hv$, where $0 \leq h = \frac{a+c+d}{1-(b+d)} < 1$.

Example 2.3. $F(t_1, ..., t_5) = t_1 - k\max\{t_2, t_3, t_4, t_5\}$, where $k \in \left[0, \frac{1}{2}\right)$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, u+v) = u - k(u+v) \leq 0$ which implies $u \leq hv$, where $0 \leq h = \frac{k}{1-k} < 1$.

Example 2.4. $F(t_1, ..., t_5) = t_1 - k\max\left\{t_2, t_3, \frac{t_4+t_5}{2}\right\}$, where $k \in [0, 1)$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, u+v) = u - k\max\left\{u, v, \frac{u+2v}{3}\right\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < 1$.

Example 2.5. $F(t_1, ..., t_5) = t_1^2 - at_2t_3 - bt_3t_4 - ct_4t_5$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, u+v) = u^2 - auv - buv - cv(u+v) \leq 0$. If $u > v$, then $u[1-(a+b+2c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a+b+2c} < 1$.

Example 2.6. $F(t_1, ..., t_5) = t_1 - at_2 - b\max\{2t_3, t_4 + t_5\}$, where $a, b \geq 0$ and $a + 3b < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, u+v) = u - av - b\max\{2v, u+2v\} \leq 0$. If $u > v$, then $u[1-(a+3b)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + 3b < 1$.

Example 2.7. $F(t_1, ..., t_5) = t_1 - at_2 - b\max\{t_3 + t_4, 2t_5\}$, where $a, b \geq 0$ and $a + 4b < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, u, v, u+v) = u - av - b\max\{u+v, 2(u+v)\} = u - av - 2b(u+v) \leq 0$. Hence $u \leq hv$, where $0 \leq h = \frac{a+2b}{1-2b} < 1$.

Example 2.8. $F(t_1, ..., t_5) = t_1^2 - at_2^2 - bt_3^2 - ct_4t_5$, where $a, b, c \geq 0$ and $a + b + 2c < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u, v, u+v) = u^2 - av^2 - bu^2 - cv\,(u+v) \leq 0$. If $u > v$, then $u^2[1 - (a + b + 2c)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a + b + 2c} < 1$.

Example 2.9. $F(t_1, ..., t_5) = t_1 - a\max\{t_2, t_3\} - b\max\{t_4, t_5\}$, where $a, b \geq 0$ and $a + 2b < 1$.

(F_2) : Let $u, v \geq 0$ be and $F(u, v, u, v, u+v) = u - a\max\{u, v\} - b\,(u+v) \leq 0$. If $u > v$, then $u[1 - (a + 2b)] \leq 0$, a contradiction. Hence $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + 2b < 1$.

3. Main results

Theorem 3.1. *Let (X, Gp) be a Gp - complete metric space and $\{T_n\}_{n\in\mathbb{N}} : (X, Gp) \to (X, Gp)$ be a sequence of mappings such that for all $x, y, z \in X$ and $i, j, k \in \mathbb{N}$:*

$$F(Gp(T_ix, T_jy, T_kz), Gp(x, y, z), Gp(T_ix, y, T_kz), Gp(T_ix, z, T_jy), Gp(T_jy, T_kz, x)) \leq 0 \tag{3.1}$$

where $F \in \mathfrak{F}_{Gp}$. Then, $\{T_n\}_{n\in\mathbb{N}}$ has a unique common fixed point.

Proof. Let x_0 be any arbitrary point of X. We define a sequence $\{x_n\}$ in S such that $x_{n+1} = T_{n+1}x_n$, $n = 0, 1, 2, ...$.

By (3.1) we have successively

$$F(Gp(T_nx_{n-1}, T_{n+1}x_n, T_{n+2}x_{n+1}), Gp(x_{n-1}, x_n, x_{n+1}), Gp(T_nx_{n-1}, x_n, T_{n+2}x_{n+1}), Gp(T_nx_{n-1}, x_{n+1}, T_{n+1}x_n), Gp(T_{n+1}x_n, T_{n+2}x_{n+1}, x_{n-1})) \leq 0$$

$$F(Gp(x_n, x_{n+1}, x_{n+2}), Gp(x_{n-1}, x_n, x_{n+1}), Gp(x_n, x_n, x_{n+2}), Gp(x_n, x_{n+1}, x_{n+1}), Gp(x_{n+1}, x_{n+2}, x_{n-1})) \leq 0. \tag{3.2}$$

By (Gp_2),

$$Gp(x_n, x_n, x_{n+2}) \leq Gp(x_n, x_{n+1}, x_{n+2})$$

and

$$Gp(x_{n-1}, x_n, x_n) \leq Gp(x_{n-1}, x_n, x_{n+1}).$$

By (Gp_4) and (Gp_2)

$$\begin{aligned} Gp(x_{n-1}, x_{n+1}, x_{n+2}) &\leq Gp(x_{n-1}, x_n, x_n) + Gp(x_n, x_{n+1}, x_{n+2}) \\ &\leq Gp(x_{n-1}, x_n, x_{n+1}) + Gp(x_n, x_{n+1}, x_{n+2}). \end{aligned}$$

By (3.2) and (F_1) we obtain

$$F(Gp(x_n, x_{n+1}, x_{n+2}), Gp(x_{n-1}, x_n, x_{n+1}), Gp(x_n, x_{n+1}, x_{n+2}), Gp(x_{n-1}, x_n, x_{n+1}), Gp(x_{n-1}, x_n, x_{n+1}) + Gp(x_n, x_{n+1}, x_{n+2})) \leq 0.$$

By (F_2) we obtain

$$Gp(x_n, x_{n+1}, x_{n+2}) \leq hGp(x_{n-1}, x_n, x_{n+1})$$

which implies

$$Gp(x_n, x_{n+1}, x_{n+2}) \leq h^n Gp(x_0, x_1, x_2). \tag{3.3}$$

Now for any integers $k \geq m \geq n \geq 1$ we obtain by (Gp_4) that

$$\begin{aligned} Gp(x_n, x_m, x_k) &\leq Gp(x_n, x_{n+1}, x_{n+2}) + Gp(x_{n+1}, x_{n+2}, x_{n+3}) + ... + \\ &+ Gp(x_{k-2}, x_{k-1}, x_k) \\ &\leq h^n \left(1 + h + ... + h^{k-n}\right) Gp(x_0, x_1, x_2) \\ &\leq \frac{h^n}{1-h} G(x_0, x_1, x_2) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since by (Gp_2), $Gp(x_n, x_m, x_m) \leq Gp(x_n, x_m, x_k)$ it follows that $Gp(x_n, x_m, x_m) \to 0$ as n, $m \to \infty$ and thus, $\{x_n\}$ is a Gp - Cauchy sequence. Since (X, Gp) is a Gp - complete metric space, by Theorem 1.5, (3.3) and Definition 1.4, there exists $u \in X$ such that $\lim_{n,m\to\infty} Gp(x_n, x_m, x_m) = \lim_{n\to\infty} Gp(u, x_n, x_n) = Gp(u, u, u) = 0$.

Now we prove that u is a common fixed point of $\{T_n\}_{n\in\mathbb{N}}$.

By (3.1) we have successively

$$F(Gp(T_n x_{n-1}, T_j u, T_k u), Gp(x_{n-1}, u, u), Gp(T_n x_{n-1}, u, T_k u), \\ Gp(T_{n-1} x_{n-1}, u, T_j u), Gp(T_j u, T_k u, x_{n-1})) \leq 0,$$

$$F(Gp(x_n, T_j u, T_k u), Gp(x_{n-1}, u, u), Gp(x_n, u, T_k u), \\ Gp(x_n, u, T_j u), Gp(T_j u, T_k u, x_{n-1})) \leq 0. \tag{3.4}$$

Letting n tends to infinity we obtain

$$F(Gp(x_n, T_j u, T_k u), 0, Gp(u, u, T_k u), \\ Gp(u, u, T_j u), Gp(u, T_j u, T_k u)) \leq 0.$$

By (Gp_2) and (F_1) we obtain

$$F(Gp(u, T_j u, T_k u), Gp(u, T_j u, T_k u), Gp(u, T_j u, T_k u), \\ Gp(u, T_j u, T_k u), Gp(u, T_j u, T_k u) + Gp(u, T_j u, T_k u)) \leq 0.$$

By (F_2) it follows that

$$Gp(u, T_j u, T_k u) \leq h Gp(u, T_j u, T_k u)$$

which implies

$$Gp(u, T_j u, T_k u) = 0.$$

By Lemma 1.5 (1), $u = T_j u = T_k u$. Thus, u is a common fixed point of $\{T_n\}_{n\in\mathbb{N}}$.

Suppose that $\{T_n\}_{n\in\mathbb{N}}$ has another common fixed point v.

Then by (3.1) we have successively

$$F(Gp(T_i u, T_j u, T_k v), Gp(u, u, v), Gp(T_i u, u, T_k v), \\ Gp(T_i u, v, T_j u), Gp(T_j u, T_k v, u)) \leq 0,$$

$$F(Gp(u, u, v), Gp(u, u, v), Gp(u, u, v), \\ Gp(u, v, v), Gp(u, v, v)) \leq 0.$$

By (F_1) we have

$$F(Gp(u,u,v),Gp(u,u,v),Gp(u,u,v),\\ Gp(u,u,v),Gp(u,v,v)+Gp(u,u,v))\leq 0.$$

By (F_2) we have

$$Gp(u,u,v)\leq kGp(u,v,v),$$

which implies

$$G(u,v,v)=0.$$

By Lemma 1.5 (1), $u=v$.

Hence, u is the unique common fixed point. □

Theorem 3.2. *Let (X,Gp) be a Gp - complete metric space and $\{T_n\}_{n\in\mathbb{N}}:(X,Gp)\to(X,Gp)$ be a sequence of mappings such that for all $x,y,z\in X$ and $i,j,k\in\mathbb{N}$:*

$$F(Gp(T_ix,T_jy,T_kz),Gp(x,y,z),Gp(T_ix,y,z),\\ Gp(x,T_jy,z),Gp(x,y,T_kz))\leq 0 \tag{3.5}$$

where $F\in\mathfrak{F}_{Gp}$. Then, $\{T_n\}_{n\in\mathbb{N}}$ has a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 3.1. □

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